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TEXT-BOOK

ON

HYDRAULICS

BY

GEORGE E. RUSSELL

PROFESSOR OF HYDRAULICS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

THIRD EDITION

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WITHDRAWN

To all those, acclaimed or unknown, who since the beginning of science have with zeal and patience sought the development and dissemination of a knowledge of the laws of hydraulic phenomena, this modest volume is appreciatively dedicated.

PREFACE

This book, the first edition of which appeared in 1909, was originally intended to serve as an elementary text for class-room use. It therefore differed in character from the several excellent reference books which then existed, being unusually brief and treating only the simpler and more fundamental principles of hydraulics. No discussion of the application of the principles to hydraulic machinery, or hydraulic engineering in general, was given. It seemed that this field was too broad for inclusion in such a book. An attempt was made to so arrange the material that its order might be clear and logical, and its usefulness limited only by the ability of the instructor.

That a demand for such a text existed, was shown by its immediate adoption by many of the technical schools of this country, and by the fact that its total printing has reached into many thousand copies. Since its first publication, several other texts similar in character have appeared indicating a preference for short texts. There seems to have been no such general addition to the list of reference books, and no general collation of recent available experimental data has been attempted. Students leaving the class-room for the field of practice have found few continuing books suitable for their needs. It occurred to the author that the revision of the original text might be attempted, and such additions made as would allow its use by the engineer as a reference book, and not impair its value as a student text. To this end the entire book has been rewritten. The subject-matter of each chapter has been made to include the result of more recent research work and in several instances entirely new material appears. The chapter on Flow in Pipes has necessarily been greatly expanded so as to include important available data on this most common engineering subject. Many tables of coefficients have been added and the list of problems at the end of each chapter increased.

It is the hope of the author that the efforts of this new edition may be justified and that the teacher, student and engineer will find in the new book all that was meritorious as well as all that was desirable, but absent, in the old.

G. E. R.

Cambridge, Massachusetts,
March, 1925.

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HYDRAULICS

CHAPTER I

INTRODUCTION

1. Definitions.—A fluid body is a mass of particles among which the force of cohesion is so slight that they may be easily made to change their relative positions by the application of small forces. We may conceive of a body as being entirely devoid of internal friction, thus permitting perfect freedom of motion of one particle past another. Between two particles the action of any force must, in this case, be in a direction *normal* to the contact surfaces, and the force cannot have any *tangential* component. Such a body we will call a "*Perfect Fluid*," although it is not known that one exists. Air, gases, water, alcohol, mercury and many other bodies possess fluidity to a high degree and are often spoken of as perfect fluids. The extent to which the tangential component is present determines the ease with which the fluid may be made to change its shape or to flow. The tangential component is, therefore, a measure of the fluid's *viscosity*. Oils, glycerine, molasses, etc., represent fluids having different degrees of viscosity.

Fluids may be either gaseous or liquid. The general study of the principles and laws which control the behavior of liquid bodies, when at rest or in motion, is generally referred to as "*Hydraulics*," although the derivation of the word signifies the flow of water in a pipe, and thus implies motion. There are naturally two divisions of the subject, *Hydrostatics*, a discussion of liquids at rest, and *Hydrokinetics*, a study of liquids in motion.

It is to be noted here that, while we shall confine ourselves in future discussions to the one liquid, water, the same laws and reasonings can be applied to any liquid whose physical properties resemble those of water. For liquids at rest, the governing laws are independent of the degree of fluidity, since *motion* of the particles must take place before the tangential stresses can be

developed. In the case of liquids in motion, this independence does not hold.

2. Physical Properties of Water.—Although water is not a perfect fluid, it is permissible for all engineering purposes to so assume it whenever its particles are relatively at rest. In all problems, therefore, dealing with static conditions, the mutual pressures between individual particles will be normal to the contact surfaces; and against any plane or surface in the water the pressure will be *normal.*

Weight.—The weight of water per unit volume varies with its temperature and purity. As one would expect, water near the boiling point is much lighter than at 39.3° Fahr., where it has its maximum density. Salt water, too, is known to be heavier than fresh, although the difference is not great. In the following table is given the weight of a cubic foot of pure water at various degrees of temperature as determined by Rossetti.

Weight of Distilled Water

Temperature Fahrenheit.	Pounds per Cubic Foot.	Temperature Fahrenheit.	Pounds per Cubic Foot.	Temperature Fahrenheit.	Pounds per Cubic Foot.
32	62.416	95	62.061	160	61.006
35	62.421	100	61.998	165	60.904
39.3	62.424	105	61.933	170	60.799
45	62.419	110	61.865	175	60.694
50	62.408	115	61.794	180	60.586
55	62.390	120	61.719	185	60.476
60	62.366	125	61.638	190	60.365
65	62.336	130	61.555	195	60.251
70	62.300	135	61.473	200	60.135
75	62.261	140	61.386	205	60.015
80	62.217	145	61.296	210	59.893
85	62.169	150	61.203	212	59.843
90	62.118	155	61.106		

It will be noticed that for ordinary ranges of temperature the weight per cubic foot is not far from 62.4 lb.; and if we make some allowance for the presence of impurities, an average value of 62.5lb. will suffice for ordinary calculations. In testing motors and pumps, where it is necessary to make accurate measurements of volume and weight, it is well to select from the above-mentioned

table a unit weight to accord with the observed temperature of the water. For calculations involving salt water a mean value of 64.0 lb. may be assumed.

In subsequent formulæ in this book, the weight per cubic foot of water will be represented by w.

Compressibility.—Water has practically perfect elasticity of volume, but no elasticity of shape. It suffers sensible compression under great pressures, but regains its original volume upon removal of the pressure. Grassi found that a pressure of 14.7 lb. per square inch (one atmosphere) on each of the faces of a cube of water caused it to lose about 0.000050 of its original volume when its temperature was 32° Fahr. Increases in temperature result in an increase in resistance to being compressed. For pressures below 1000 lb. per square inch, experiments indicate that the amount of compression varies directly as the pressure (temperature constant); but for higher pressures this law no longer holds. By using a pressure of 65,000 lb. per square inch, Hite obtained a reduction in volume amounting to approximately 10 per cent. Had Grassi's coefficient of reduction remained constant, Hite would have obtained more than a 20 per cent. reduction. Since hydraulic pressures are ordinarily less than 1000 lb. per square inch intensity, we may proceed to calculate reduction in volume on the assumption that for each atmosphere of pressure (14.7 lb. per square inch), a reduction of volume equal to one-twenty thousandth (0.00005) of the original volume will take place.

Example.—Assuming a cubic foot of sea water to weigh 64.0 lb. per cubic foot at the sea level, determine its unit weight at a depth of 500 ft.

Let w = unit weight at the surface.
w' = unit weight at 500 ft. depth.
V = volume of a given mass at surface.
V' = volume of same mass at 500 ft. depth.

The unit weight will vary inversely as the volumes, or

$$\frac{w'}{w} = \frac{V}{V'}.$$

By the law of Grassi

$$V' = (V - 0.00005V \times \text{pressure in atmospheres}).$$

At a depth of 500 feet it will be shown later that the pressure is approximately $\frac{500}{33.1}$ atmospheres (here 33.1 represents the depth of salt water necessary to give a pressure of one atmosphere). We may therefore write

$$w' = \frac{w \times V}{V - 0.00005\ V \times \frac{500}{33.1}},$$

or, since V in this particular example is 1 cubic foot,

$$w' = \frac{64}{1 - \frac{0.025}{33.1}} = 64.05 \text{ lb. per cubic foot.}$$

It may be noted that this figure is obtained under the assumption that an increase in pressure of one atmosphere follows each 33.1 foot increment in depth, whereas in reality, the slight increase in weight per cubic foot would cause this increment to gradually diminish as we descend from the free surface. The resulting error, however, is too slight to make a notable difference in the result.

Again, assuming Grassi's law to hold up to a pressure intensity of 1000 lb. per sq. inch we will find the diminution in volume resulting from increasing the pressure from atmospheric to this pressure to be

$$\frac{(1000-14.7)}{14.7} \times \frac{1}{20000} = .00335$$

or about one-third of one per cent. The increase in weight per unit of volume would be the same.

From both of the above computations it may be seen that increase in unit weight due to compression is too slight to be regarded in ordinary computations. The basis for the popular belief that water is incompressible is also apparent.

Modulus of Elasticity.—On the basis of Grassi's figures we may determine a modulus of elasticity for water which will be a *volume* or *bulk* modulus. It is the value of the ratio of the stress per unit of area to the change per unit of volume. Denoting the former by p and the latter by $\Delta V \div V$, where ΔV represents the total change in volume, we have

$$E = \frac{p}{\Delta V \div V} = \frac{14.7}{.00005} = 294{,}000 \text{ lb. per square inch.}$$

It should be noted that this value of E holds only for pressures under 1000 lb. per square inch intensity and for temperatures around freezing. Increase in temperature raises the value of E slightly. For ordinary temperatures E will have a value of approximately 300,000 lb. per square inch. This is one one-hundredth of the value of E for mild steel and hence water may be said to be one hundred times more compressible than steel when the latter is stressed below the elastic limit.

3. Equal Transmission of Pressure.—One of the marked characteristics of a fluid is its ability to transmit pressure, applied to its surface, equally in all directions and with undiminished intensity. Thus in the accompanying figure if a pressure of 10 lb. per square inch be applied to the water surface *ab* by means of a force exerted upon the piston, then this pressure will exist in *all parts* of the contained liquid and upon all parts of the reservoir walls in addition to any pressure that may previously have existed by virtue of the weight of the water itself.

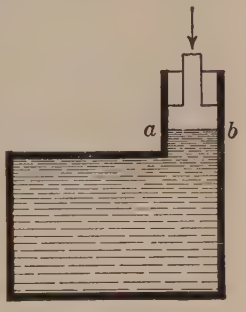


FIG. 1

This principle finds an application in the hydraulic press. Here a comparatively small force, existing on the face of a piston in a small cylinder filled with water, is transmitted by a pipe to a larger cylinder, where it acts upon the face of a larger piston and exerts a total force in proportion to the face area. The ratio of this total force to that acting on the small piston will be as the ratio of the respective piston areas.

4. Units of Measure.—The elements of distance, force and time enter into all hydraulic discussions, and will be measured by units of the foot-pound-second system. The foot is the common English foot of 12 inches, and the pound is the avoirdupois pound of 16 ounces. Unless otherwise stated, these will be the units employed in all hydraulic formulæ found in this work, and it is essential to see that all data are reduced to these units before making substitution in the formulæ. In accordance with this

plan, areas will be expressed in square feet; volumes, in cubic feet; intensity of pressure, in pounds per square foot; and duration of time, in seconds. It is quite common practice, however, to measure intensity of pressure in pounds per square *inch*, and volumes are often given in gallons. In this connection the following relations will be found useful:

$$1 \text{ cu. ft.} = 7.48 \text{ U. S. gal.}$$
$$1 \text{ U. S. gal.} = 231 \text{ cu. in.}$$
$$1 \text{ U. S. gal.} = 0.8331 \text{ Imperial gal.}$$

The Imperial gallon is mostly used in Great Britain.

As the motion of water, in most cases, is due to the action of gravity, the quantity g will often appear in this work. By g we represent the acceleration, due to gravity, of a freely falling body. It is the change in velocity during one second of time, and consequently will be expressed in feet per second, per second. Its value at any point on the earth's surface depends on the distance of the point from the center of the earth's mass, and may be obtained from the following formula by Pierce:

$$g = 32.0894\ (1 + 0.0052375 \sin^2 l)\ (1 - 0.0000000957\, e).$$

Here l is the latitude of the place and e its distance in feet above sea level. A mean value for g, and one sufficient for all ordinary calculations, may be stated as 32.2 ft. per second, per second.

5. Numerical Problems.—In order to familiarize one's self with the principles of hydraulics, no better way can be found than in the working of many numerical problems. In fact it is the only way in which a firm grasp on the subject can be attained. To this end numerous problems will be given in this book, and should be worked out by the student. Habits of neatness, accuracy and method should be acquired in their solution, and it is desirable that each problem be done in ink on a separate sheet of good paper, and finally bound into one convenient volume. Such a book will prove of great value to the student, not only in the training received while making it, but for reference in after years.

In general the use of logarithms is recommended, tables of four or five places being sufficiently accurate. The student is warned against deceiving himself as regards the accuracy of his results, and he should inspect his data with a view to deter-

mining the probable percentage of error in his answer. This latter should not be given so as to indicate a higher degree of accuracy than is warranted by the data. A problem may serve to illustrate the point. The cross-section of a natural stream is figured as 756.2 sq. ft., and the mean velocity of the stream past the section is 3.2 ft. per second as obtained by observations with a current meter. The discharge of the stream as given by the product of the above figures is 2419.84 cu. ft. per second. This result, however, is misleading, as the area may be in error by 10 sq. ft., and the figure for velocity in error by at least 0.2 ft. per second. If the area were 745 sq. ft. and the velocity 3.0 ft. per second, the rate of discharge would be 2235 cu. ft. per second, while with an area of 765 sq. ft. and a velocity of 3.4 ft. per second, the discharge would be 2601 cu. ft. per second. Between these two figures there is a wide difference, and our answer contains all the possible accuracy when stated as 2400 cu. ft. per second.

The use of a slide rule will often save laborious calculations, and the student should early acquire a knowledge of its use and limitations.

PROBLEMS

1. If a body of water has a temperature of 32° F., what decrease in volume will follow the application to it of a pressure of 588 lb. per square inch? What would be its initial and final unit weight if pure?

2. What is the approximate unit weight of sea water at a depth of 5 miles if 64.0 lb. per cu. ft. be assumed its unit weight at the surface?

3. By the use of the table in Art. 2, determine the increase in a volume of pure water, which at 50° F. contains 120 cu. ft., resulting from raising its temperature through 100° F.

4. A hydraulic press (Art. 3) contains 30 cu. ft. of water at 32° F. The areas of the small and large pistons are ¼ square inch and 100 square inches respectively. Considering only the change in water volume due to pressure, how much motion of the small piston will follow the application to it of a force of 500 lb., assuming the large piston to move through a distance of ½ inch?

5. If a vertical, cylindrical column of water contained in a standpipe is 100 ft. high, what would be its height (cross-section remaining constant) if water were absolutely incompressible? (Temperature 32° F.)

CHAPTER II

HYDROSTATICS

6. Intensity of Pressure.—By intensity of pressure we denote the pressure per unit of area. If dA represents a very small part of an area A, and dP the total pressure upon it, then the intensity of pressure is

$$p = \frac{dP}{dA}.$$

If the intensity be the same for the entire area A, we may write the total pressure on A as

$$P = \int p.\, dA = pA,$$

from which

$$p = \frac{P}{A}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The student will see that if p is not the same over the entire area, formula (1) gives an *average* value of p. As has been already pointed out, its numerical value in hydraulic work is commonly expressed in pounds per square inch.

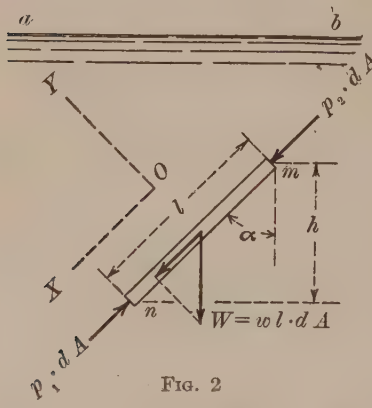


FIG. 2

7. General Relation between Pressures at Different Points in a Liquid.—In Fig. 2 let m and n be any two points arbitrarily chosen in the liquid whose upper or "free" surface is ab. Between m and n is shown any imaginary prism of water whose end faces contain the points in question. The cross-sectional area dA of the prism being infinitely small, the pressure on these faces may be assumed as of uniform intensity. Denoting these intensities by p_1 and p_2, the total end pressures are $p_1 dA$ and $p_2 dA$ respectively.

The other forces acting upon the prism are the pressures on its side faces and its own weight, which may be written $w.l.dA$. Since all these forces form a balanced system, they may be resolved into components along rectangular axes and the algebraic sum of each set put equal to zero. With axes as shown in Fig. 2, we may then write (remembering that the side pressures have no X comps.):

$$\Sigma X = p_1 dA - p_2 dA - w \,.\, l \,.\, dA \cos \alpha = 0$$

from which

$$p_1 - p_2 = wh. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

h being equal to $l \cos \alpha$, the vertical distance between m and n. From (2) we see that the difference in pressure is dependent solely upon the difference in elevation between the two points.

Note.—*If the points lie in the same horizontal plane, then the pressures are the same.*

Had the point m been chosen in the surface ab, p_2 would have been the pressure of the atmosphere above the liquid, or p_a, and for the value of the pressure at any point lying a distance h below the surface, we should have

$$p = p_a + wh. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The appearance of p_a in this expression is an illustration of the principle of free transmission of pressure.

8. Illustration of the Foregoing Principles.—In Fig. 3 the reservoir M is connected by the tube CD with the reservoir at

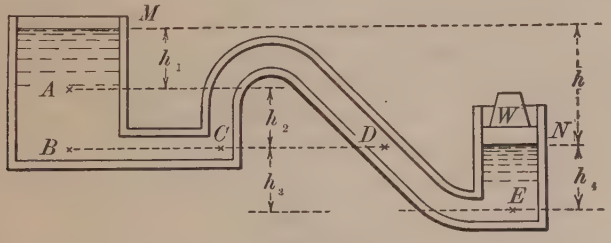


FIG. 3

the lower level N. All portions of the tube are assumed filled with water.

The weight W on the piston at N is just heavy enough to main-

tain the water in M at a constant level. For the pressure at any point such as A we may write

$$p_1 = p_a + wh_1,$$

and for a point B at a depth h_2 below A the pressure is evidently p_1 increased by wh_2.

$$\therefore p_2 = p_1 + wh_2.$$

This value of p_2 must hold good for points C and D as they lie in the same horizontal plane. For a point E we have

$$p_3 = p_2 + wh_3,$$

and for the pressure at N, beneath the piston,

$$p_4 = p_3 - wh_4,$$

which may be written, by aid of the previous equations,

$$p_4 = p_a + wh_1 + wh_2 + wh_3 - wh_4,$$

or

$$= p_a + w(h_1 + h_2 + h_3 - h_4)$$

or

$$p_4 = p_a + wh.$$

This last might have been at once derived from equation (3), since h is the distance of the point at N below the *free* water surface.

9. Relative and Absolute Pressure. Pressure Head.—In the previous problem, p_4 is the pressure on each unit area of the under side of the piston at N, which is open to atmosphere on its upper side. It is clear that, as far as the effectiveness of the water pressure in sustaining the weight W is concerned, p_a need not enter into the discussion, as it acts on *both* sides of the piston and produces no resultant force. Eliminating p_a, we obtain a value for p_4 which measures the excess of the pressure over, or above, the atmospheric. We call this *Relative Pressure,* or *Hydrostatic Pressure.*

If p_a be *included* in p_4, the pressure is measured above vacuum, or absolute zero, and is called *Absolute Pressure.* This latter need be seldom used in the simple problems of statics, as atmospheric pressure generally appears on a body as a set of balanced forces which produce only *internal* stress and may be discarded.

Thus, if we are finding the pressure against a reservoir wall, we are not concerned with the absolute pressure, since the atmosphere is acting on the opposite face of the wall also, and has no effect upon its stability.

Equation (3), therefore, becomes, for most cases (using "relative" units),

$$p = wh. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In either formula (3) or (4) it should be remembered that, in substituting numerical values, care must be taken to see that the units used are consistent. If w be in pounds per cubic feet, then h must be measured in feet and p will result in pounds per square foot. Similarly h may be in meters; w, the weight, in kilograms per cubic meter; and p will be in kilograms per square meter. The formula is therefore homogeneous, admitting the use of any system of units.

Example.—Let it be desired to find the intensity of pressure at a depth of 75 feet in fresh water.

$$p = wh = 62.5 \times 75 = 4680 \text{ lb. per square foot.}$$

In order to avoid the use of large figures, it is quite common in practice to express hydraulic pressures in lb. per *square inch*, in which case the above may be written

$$p = \frac{62.5 \times 75}{144} = \frac{75}{2.3} = 32.6 \text{ lb. per square inch.}$$

That is, to find the pressure in *pounds per square inch* corresponding to any *head in feet*, divide the head by 2.3. This is a very convenient rule and points out the fact that 2.3 feet of water causes a pressure of 1 lb. per square inch.

Pressure Head.—The quantity h has been used to represent the vertical distance from the free surface to the point in question. The distance is commonly called the *head* on the point, and because it causes the pressure p, it is also known as the *pressure head.* The same name is applied to its equivalent, $\frac{p}{w}$, and it is advisable to at once become familiar with this way of expressing pressure head inasmuch as it constantly occurs in this form when discussing problems of fluid flow.

If we are using *absolute* pressure units, the pressure head will be $(h + 34)$ ft., since atmospheric pressure is equivalent to that

produced by a head of water measuring 34 ft. (See Art. 11.) Whenever the pressure (p_o) on the free surface varies from the atmospheric, the pressure head in absolute units will be the sum of the hydrostatic head and that corresponding to the pressure p_o. That is,

$$\frac{p}{w} = \left(h + \frac{p_o}{w}\right).$$

10. Definition and Form of a Free Surface.—We have already referred to the water surface in contact with the atmosphere as a "free" surface. Properly speaking, if a liquid has a free surface, then upon that surface there is *no pressure.* It is common, however, to regard a surface upon which there is only atmospheric pressure as a free surface.

It is at once evident that a free surface is one of "equal pressure" and hence is horizontal (Art. 7).

11. Atmospheric Pressure and Water Barometer.—The atmosphere is a perfect fluid, exerting a normal pressure upon all surfaces with which it is in contact. It has been experimentally found that the intensity of its pressure, under normal barometer conditions at sea level, is about 14.7 lb. per square inch. Were it an incompressible fluid, we should find the pressure at a point distant h feet above sea level diminished by $w'h$, w' denoting the weight of atmosphere per cubic unit. Assuming this latter at 0.0807 lb., an experimental value found at freezing temperature under a pressure of 14.7 lb. per square inch, the pressure at an elevation of 6200 ft. above sea level would be

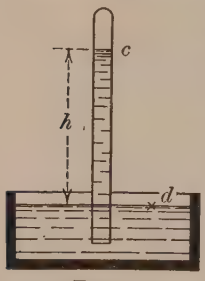


FIG. 4

$$p = 14.7 - \frac{0.0807 \times 6200}{144} = 11.2 \text{ lb. per square inch.}$$

Actually the pressure is a little greater than this (11.6 lb. p. s. in.), owing to the variation in density of the atmosphere.

Water Barometer.—This very simple device may be used to demonstrate the existence of atmospheric pressure and also to approximately measure its intensity. A long glass tube, closed at one end, is completely filled with water and the open end temporarily stopped. It is then placed in an inverted position with

the stopped end beneath a water surface and the stopper removed. If the length of the tube be as great as 35 feet, or more, it will be found that the water remains partly filling it, there being a portion of the tube, above the point c, (Fig. 4) which contains a *partial* vacuum. Were the vacuum perfect the pressure at c would be absolute zero, while at d it would be 14.7 lb. per square inch. From equation (2) we have

$$p_d - p_c = wh$$
$$(14.7 \times 144) - 0 = 62.5\,h$$
$$h = 33.9 \text{ feet.}$$

This is the height to which a water column may be maintained by ordinary atmospheric pressure and the value of 34 feet is commonly used. The above computation assumes a perfect vacuum in the tube above the point, c. As a matter of fact the space is usually filled with a water vapor which forms at even ordinary temperatures and tends to decrease the height of the water column. The formation of the vapor is simply due to evaporation which is influenced by temperature. The pressure-head, in feet of water, produced by the vapor at different temperatures, is shown in the accompanying table.

Vapor—Pressure-Temperature Table for Water Vapor

Temperature, Fahr.	$\frac{p_v}{w}$	Temperature, Fahr.	$\frac{p_v}{w}$	Temperature, Fahr.	$\frac{p_v}{w}$
−20°	0.02	60°	0.59	140°	6.63
−10	.03	70	0.83	150	8.54
0	.05	80	1.16	160	10.90
10	.08	90	1.59	170	13.78
20	.13	100	2.17	180	17.28
30	.19	110	2.91	190	21.49
40	.28	120	3.87	200	26.52
50	.41	130	5.09	212	33.84

If mercury be used in the tube of Fig. 4, instead of water, the height of the column will be

$$h = \frac{p_d - p_c}{w'} = \frac{(14.7 \times 144 - 0)}{848.7}$$
$$= 2.5 \text{ ft. or } 30 \text{ in.,}$$

and the tube can be made much shorter and more easy to handle. The height of the mercury will not be affected by the formation of a vapor at the top of the column, save at high temperatures.

When suction pumps are placed above the level of the water which supplies them, their operation depends upon the pressure of the atmosphere which raises a column of water within the suction pipe as soon as the pump has created a partial vacuum there. Were the vacuum perfect, the limit to the height at which the pump could be set above the supply-water level would be 34 feet. The impossibility of a perfect vacuum, however, coupled with the frictional resistance offered the moving water by the walls of the suction pipe, causes the distance to be materially reduced. The best of this type of pump can rarely exceed a lift of 28 feet and a good working-lift is from 20 to 25 feet.

12. Water Piezometer.—A piezometer, as the name implies, is an instrument for the measuring of pressures. A water piezometer generally consists of a straight glass tube inserted in the side of a reservoir, containing water under pressure, and carried vertically upward to a height sufficient to prevent overflow (Fig. 5). The height h of the free surface in the tube above any point m in the reservoir exactly measures the pressure at that point, since $p = wh$. It is quite obvious that the location of the point of insertion makes no difference in the height h to which water will rise in the tube and ab marks the level of all such piezometer columns. As has been noted previously, the absolute head on the point m is $h + 34$ ft. since the top of the tube is open to the atmosphere. This form of piezometer is found very useful in experimental work, as the tube may be placed in any convenient position and connected with the reservoir by means of a piece of rubber tubing. Care should be taken in measuring small pressures that the internal diameter of the tube be large enough to prevent capillary action from affecting the height h. Gibson in his "Hydraulics" recommends the use of tubes having an internal diameter greater than 0.3 in., and figures that for this particular size the effect of capillarity is to increase the true height of the column by 0.15 in. (water at 68° Fahr.). For a 0.5-in. tube the increase is

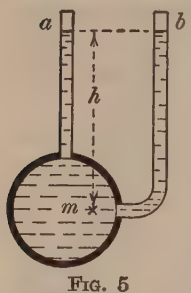


FIG. 5

0.092 in., and it would seem that no tube of smaller diameter should be used for accurate results.

If the pressure in the reservoir (Fig. 5) be maintained at less than the atmospheric, no column will rise in the piezometer and air will enter continuously at the top. If the top be bent over and downward into a vessel of water (Fig. 6), the atmosphere will cause a column of the water to rise to a height h in the tube, from which a measure of the pressure is obtained. Neglecting the weight of the air caught in the portion ab of the tube, the pressure on the free surface in the reservoir is the same as that at c. This latter, from (2), we know to be

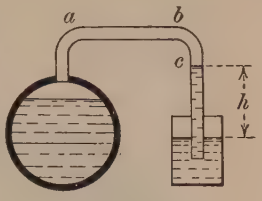


FIG. 6

$$p = p_a - wh.$$

13. The Mercurial Gauge.—It will be seen that, for pressures differing much from atmospheric, the use of the above type of gauge necessitates a long length of piezometer tube. To obviate this, mercury is generally employed and the form of the tube changed. The general arrangement is shown in Fig. 7. The tube is bent into a U-shape and into the loop of the U mercury is poured. A stopcock at c permits the expulsion of all air from the reservoir end of the tube, and the water may come into direct contact with the mercury. The difference in level z between the two mercury columns being indicative of the pressure at a, we have

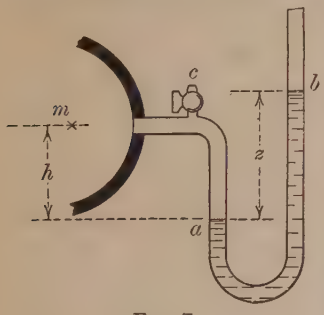


FIG. 7

$$p_m = w'z - wh,$$

or

$$p_m = w\,(13.59z - h),$$

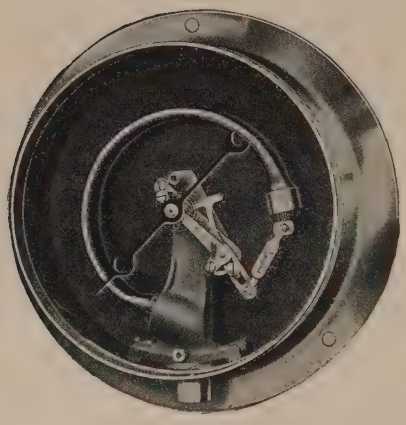
13.59 being the specific gravity of mercury, or the ratio $\frac{w'}{w}$. If the position of the U-tube is such that a is *above* m, then the negative sign in the above expression becomes positive.

Mercury may be used also with much convenience in the vacuum gauge shown in Fig. 6. For the pressure on the free surface (either in the column or reservoir) we should obtain

$$p = p_a - 13.59\,wh.$$

14. The Bourdon Gauge.—For the measurement of high pressure, use is often made of the ordinary type of steam gauge as invented by Bourdon. Its construction is such that the pressure of the fluid is communicated to a coiled tube having its inner end closed and connected by a simple rack and pinion to a hand which is free to move back and forth over a graduated dial. The pressure of the fluid tends to uncoil the tube resulting in a movement of the hand. The dial is calibrated by applying known pressures to the gauge and noting the position of the hand. The zero reading corresponds to atmospheric pressure. The accuracy of the gauge will depend on the care with which it is calibrated and, in careful experimental work it is desirable to calibrate it anew before and after using. Fig. 8 shows the gauge as manufactured by the Crosby Steam Gage & Valve Company of Boston.

A



B

Fig. 8. The Bourdon Gauge

15. The Differential Gauge.—This gauge is used to measure *differences* in pressure, and in general arrangement is similar to that shown in Fig. 9. A and B are two separate reservoirs connected by the tubing *a-b-c-d-e-f*. Mercury occupies the bottom of the U-shaped portion, and cocks at b and e permit the filling of the rest of the tube with water from the reservoirs. Because of greater pressure in A, there is a difference in level z between the two columns of mercury. From this measured difference we may proceed to calculate the difference in reservoir pressures as follows:

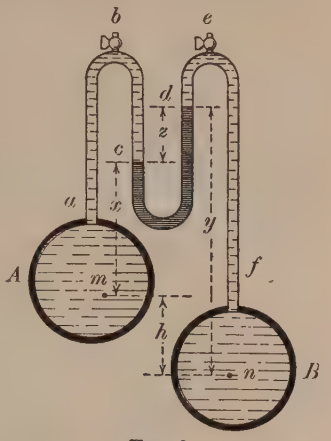


FIG. 9

At c and d the pressures are $p_c = p_m - wx$ and $p_d = p_n - wy$ respectively, m and n being points arbitrarily chosen in A and B. The difference in these pressures is measured by the mercury column z so that

$$p_c - p_d = w'z = (p_m - wx) - (p_n - wy),$$

from which we may obtain

$$p_m - p_n = w'z + w(x - y).$$

Reference to the figure shows that

$$(x - y) = -(h + z),$$

and by substitution there results

$$\begin{aligned} p_m - p_n &= w'z - w(h + z) \\ &= w\,[\,z\,(s - 1) - h], \end{aligned} \quad . \quad . \quad . \quad . \quad (5)$$

(s being the specific gravity of the mercury). If the points m and n be situated in the same horizontal plane, then $h = 0$ and (5) becomes

$$p_m - p_n = wz(s - 1) = 12.59\,wz. \quad . \quad . \quad . \quad (6)$$

This gauge finds a very wide use in practical and experimental work. Its one disadvantage, in the form as outlined, is that small differences of pressure are not possible of accurate meas-

urement, since the mercury pressure head is only one-twelfth (roughly) of the actual difference in head, $\frac{p_m}{w} - \frac{p_n}{w}$. Any slight error, therefore, in reading the head z will be magnified in the same ratio. To obviate this it is common to use, in place of the mercury, a liquid whose specific gravity is only slightly greater than that of water. Cole in some of his experiments used a mixture of carbon tetrachloride and gasoline which had a specific gravity of 1.25. From (6) it will be seen that this resulted in the relation

$$\frac{p_m - p_n}{w} = .25\ z$$

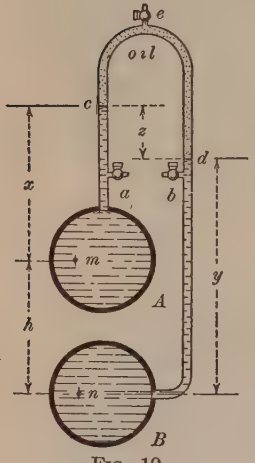


FIG. 10

or z was *four times* as great as the actual difference in pressure heads. Therefore a small error made in measuring z was rendered negligible. The liquid most commonly used in place of the mercury is kerosene oil which has a specific gravity of approximately 0.79. Since it is *lighter* than water a slight modification of the gauge is necessary (Fig. 10). Cocks at a and b permit the partial filling of the vertical tubes ac and bd. Oil is then poured in at e until the upper portion of the tube is full and all air expelled. From the difference of level z we may obtain, by a process of reasoning similar to that already used,

$$p_m - p_n = w[z(1 - s) - h].$$

$$\text{If } h = 0, \quad p_m - p_n = wz(1 - s). \quad . \quad . \quad . \quad . \quad (7)$$

Using 0.79 as the value of s for kerosene, there results

$$\frac{p_m - p_n}{w} = 0.21\ z.$$

In this case the actual difference in pressure-head is approximately one-fifth the distance, z, and small differences in pressure are readily read with minimum error.

In the heavier than water type of gauge, (Fig. 9) it is evident from equation (6) that the use of a liquid having a specific gravity of 2 results in a difference in level, z, which is just equal to the difference in pressure-heads. The distance, z, will be less or greater than the actual difference in pressure-heads, according as the specific gravity of the liquid used is greater or less than 2.

Whenever the gauge is used to measure differences in pressure in a liquid other than water, of course the value of w in all the equations of this article becomes the unit weight of the liquid, and s will be the specific gravity of the auxiliary liquid in terms of the liquid itself.

One of the most common uses of the differential gauge is to measure the fall in pressure in a given length of pipe line through which water is flowing, and thus determine the loss of head in that section. (See Arts. 38 and 90.) In this case the two stems of the gauge are connected to the pipe by means of smaller pipes or rubber tubing.

16. Total Normal Pressure on Plane Surfaces.—If a plane surface be immersed in a horizontal position, it follows from Arts. 6 and 7 that the total normal pressure on it will be

$$P = Ap = Awh,$$

A being the area of the surface. It will now be shown that for a plane surface immersed in *any* position, the total normal pressure may be calculated from

$$P = Awh_0,$$

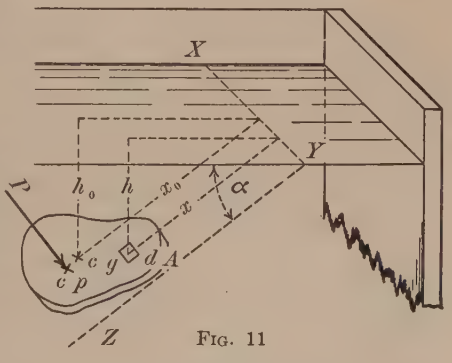


Fig. 11

if h_0 be the head on *the center of gravity* of the surface.

Figure 11 shows such a surface, lying in the plane XYZ. This plane cuts the water surface in the line XY, and the angle between the plane and the water surface we will call α. Selecting a very small part dA of the total area, so small that the pressure over it is of uniform intensity, we have, as the total pressure upon it,

$$dP = p.dA = wh.dA.$$

For total pressure on the entire area we have

$$P = \int wh.dA = \int w.x \sin \alpha.dA = w \sin \alpha \int x.dA.$$

The integral of $x.dA$ is the moment of the area about XY as an axis, and may therefore be written x_0A where x_0 is the distance from the axis to the center of gravity of the area. We may therefore write

$$P = w \sin \alpha \, x_0A,$$

or

$$P = Awh_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where h_0 is the head on the center of gravity. Hence the following theorem:

The total pressure on an immersed area is the product of that area, the weight of a cubic unit of water, and the head upon its center of gravity.

Example.—Find the total pressure on a vertical, rectangular sluice gate, 4 ft. wide by 6 ft. deep, the head on its upper edge being 10 ft.

$$P = Awh_0 = 4 \times 6 \times 62.5 \times 13 = 19{,}500 \text{ lb.} \quad \textit{Ans.}$$

17. Total Pressure on a Curved Surface.—It can be shown that the above theorem applies with equal exactness to any surface, be it plane, curved, or irregular. However, the total pressure on surfaces of the last two classes is usually of little or no practical value to the engineer. It would be the algebraic sum of a system of forces all acting in different directions, and in general we know that such a system cannot be replaced by a single resultant.

More often it is desired in the case of curved surfaces to find a *component* of normal pressure in some fixed direction. (See Art. 22.)

18. Center of Pressure.—An immersed *plane* surface is pressed upon by a system of parallel forces, infinite in number, which may be replaced by a single resultant force. The point on the surface at which this acts we will call the *Center of Pressure.* As noted in the previous paragraph, the pressures on a curved surface do not form a parallel system, hence they cannot, *in general*, be reduced to a single force.

The case of the plane surface may be represented by Fig. 11. The resultant of all the pressures on the surface is supposed to be acting at the point c. p. and we are to determine first its distance from XY, the line of intersection of the plane containing the surface and the plane of the water surface.

Considering a very small area dA, we have as the total pressure upon it,

$$dP = dA.wh \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

and its moment about XY is

$$dP.x = dA.wh.x. \quad . \quad . \quad . \quad . \quad . \quad (b)$$

If in this way the moment of the pressure on each elementary area be found, we may place their sum equal to the moment of the resultant pressure by its arm x_c. That is,

$$P.x_c = \int dP.x,$$

or, from (a) and (b)

$$x_c \int dA.wh = \int dA.wh.x. \quad . \quad . \quad . \quad . \quad (c)$$

From Fig. 11,

$$h = x.\sin \alpha,$$

which substituted in (c) gives

$$w.\sin \alpha.x_c \int x.dA = w.\sin \alpha \int x^2.dA,$$

or

$$x_c = \frac{\int x^2.dA}{\int x.dA} = \frac{I}{S}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The integral of $x^2.dA$ will be recognized as the *Moment of Inertia* of the area, while the integral of $x.dA$ is the expressed moment of the area about the chosen axis, or its *Statical Moment.*

It is evident that x_c measures only the distance to the center of pressure down from XY and does not fix it in a lateral position. To do this it is necessary to take moments about another axis (not shown in Fig. 11) lying in the plane XYZ but at right angles to the previous axis. If by y we represent the distance from any

elementary dA to this axis, we shall have as the moment of the total pressure on dA about this axis,

$$dP.y = dA.w.h.y;$$

and as before,

$$P.y_c = \int dP.y,$$

$$y_c \int dA.w.h = \int dA.w.h.y,$$

$$h = x.\sin \alpha,$$

and finally,

$$y_c = \frac{\int x.y.dA}{\int x.dA}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

In applying this formula, it is necessary that y be expressed as a function of x.

In general the engineer has only to do with the vertical position of the center of pressure and (10) is seldom used. For some special cases the lateral position is easily located, as will be shown in the following paragraphs.

In applying equation (9) it should be noted that I and S must be computed about an axis formed by the intersection of the given plane with the water surface, and that x_c measures the distance from this axis to the center of pressure.

Reference to the equation also shows that the position of the center of pressure is independent of the angle α (Fig. 11) provided it has any value other than zero. (For this value the given derivation of (9) would hold.) Hence we may vary α by swinging the surface about XY as an axis without the center of pressure changing position, provided x_0 be unchanged.

Horizontal Surfaces.—If the surface be horizontal, the head on all points being the same, the elementary pressures are all equal and the center of pressure will lie at the center of gravity.

19. Centers of Pressure for Common Figures.—The forms of surfaces most commonly met with in engineering work are the rectangle, triangle, and circle. For these we will now derive special equations for locating the center of pressure. We will assume each figure in turn to be placed in some inclined plane, XYZ, Fig. 11, and Figures 12 to 15 will be used to present views of each figure taken *normal to its own plane.*

The Rectangle. (Fig. 12.) The figure will be assumed to have one side, b, parallel to the axis XY.

From equation (9)

$$x_c = \frac{I}{S}$$

and from the principles of mechanics,

$$I = I_o + Ax_o^2,$$

I_0 representing the Moment of Inertia of the rectangle about an axis through its own center of gravity and parallel to XY.

Therefore
$$x_c = \frac{I_o + Ax_o^2}{Ax_o} = \frac{\frac{bd^3}{12} + bdx_o^2}{bdx_o}$$

$$x_c = \frac{d^2}{12x_o} + x_o$$

or
$$x_c - x_o = \frac{d^2}{12x_o} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Reference to the figure shows that $\frac{d^2}{12x_0}$ is the distance between the center of gravity and the center of pressure and, because of the simplicity of the numerical work involved, the computation of the value of this term offers the best method for locating the center of pressure. It will be noted that the center of pressure is *below* the center of gravity. If the figure lies in a vertical plane, x_o in the formula becomes h_o. If the upper edge of the rectangle lies in the water surface,

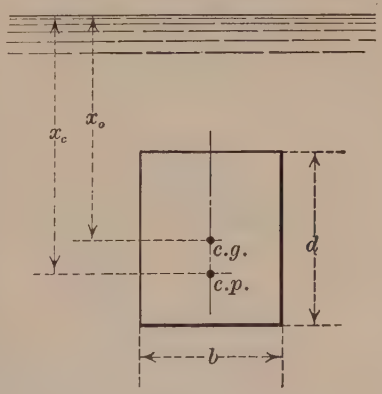


FIG. 12

$$\frac{d^2}{12x_o} = \frac{d}{6}$$

or the center of pressure lies a distance $\frac{2}{3}d$ down from the upper edge. This is a very useful fact and should be remembered as it

locates the center of pressure on the vertical faces of dams, retaining walls and other structures exposed to water pressure.

Example.—A rectangle 10 feet long by 6 feet wide lies in a plane making 45 degrees with the horizontal. Its upper 6-foot edge is 10 feet below and parallel to the water surface. Locate the center of pressure. (The reader should draw the figure.)

$$x_c - x_o = \frac{d^2}{12x_o} = \frac{10 \times 10}{12(5 + 14.14)} = 0.44 \text{ feet}$$

The Triangle. (Fig. 13.) One side will be assumed *horizontal* and the opposite vertex nearer the water surface. Proceeding as before

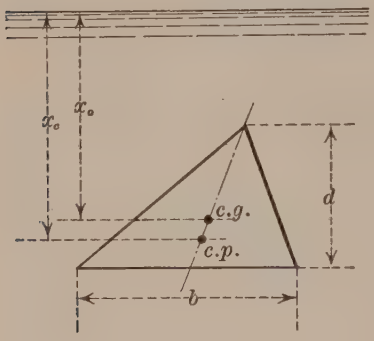


Fig. 13

$$x_c = \frac{I_o + Ax_o^2}{Ax_o} = \frac{\frac{bd^3}{36} + \frac{bd}{2}x_o}{\frac{bd}{2}x_o}$$

$$x_c = \frac{d^2}{18x_o} + x_o$$

or $$x_c - x_o = \frac{d^2}{18x_o}. \quad . \quad (12)$$

The similarity of (11) and (12) makes them easily remembered.

If the vertex lies in the water surface,

$$x_c - x_o = \frac{d^2}{18 \times \frac{2}{3}d} = \frac{d}{12}$$

$$x_c = \frac{d}{12} + \frac{2}{3}d = \frac{3}{4}d$$

The triangle may have one side horizontal but the opposite vertex *farther* from the water surface as in Fig. 14. There results no change in equation 12 as the student should verify. If in this position *the base be placed in the water surface,*

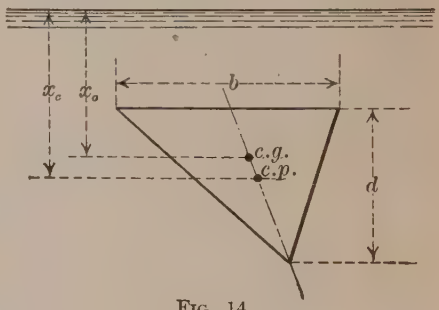


Fig. 14

$$x_c - x_o = \frac{d^2}{18\frac{d}{3}} = \frac{d}{6}$$

$$x_c = \frac{d}{6} + \frac{d}{3} = \frac{d}{2}$$

The Circle.—(Fig. 15). The diameter will be denoted by d.

$$x_c = \frac{I_o + Ax_o^2}{Ax_o} = \frac{\frac{\pi d^4}{64} + \frac{\pi d^2}{4} x_o^2}{\frac{\pi d^2}{4} x_o}$$

$$x_c = \frac{d^2}{16x_o} + x_o$$

$$x_c - x_o = \frac{d^2}{16x_o} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Lateral Position of Center of Pressure in Above Figures.—In each of the three figures just discussed, it will be seen that if we divide the area into elementary strips with lengths parallel to the water surface, each strip will be subjected to a uniform pressure and its center of pressure will lie at the middle of the strip. Since the centers of all such strips lie on a straight medial line, it follows that the resultant pressure on the entire figure lies on the medial line. In each of these figures, therefore, the center of pressure is exactly located by the formula given and the medial line.

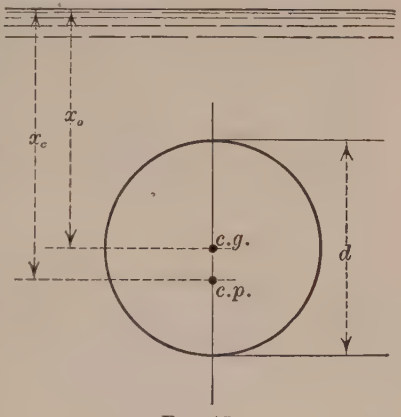


FIG. 15

Corollary.—*If an area have an axis of symmetry at right angles to the moment axis, the center of pressure will lie upon it.*

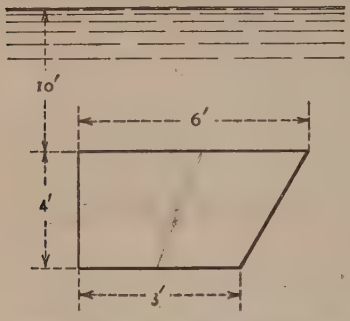
20. Center of Pressure for Irregular Figures.—In the common figures just treated we have made use of the equation

$$x_c = \frac{I_o + Ax^2_o}{Ax_o}.$$

This might have been written

$$x_c = \frac{I_o}{Ax_o} + x_o,$$

$\frac{I_o}{Ax_o}$ representing the *general* expression for $x_c - x_o$ which was specially determined in each case. The expression and method applies, of course to any plane figure, but difficulty may arise in computing I_o and locating the center of gravity. Often irregular figures (with straight sides) may be subdivided into rectangles and triangles in such a way as to make the work comparatively simple.



Example.—A trapezoid of the shape shown in the accompanying figure is immersed vertically with a head of 10 ft. on its upper edge. Find the location, both vertically and laterally, of the center of pressure.

For convenience divide the figure into a rectangle and triangle as shown, whose values of I and S may be found separately. For the rectangle,

$$I = I_o + Ax_o^2,$$

where I_0 is the moment of inertia of the figure about an axis through its center of gravity parallel to the water surface. The quantity x_0 is the distance between these two axes.

$$\therefore I = \frac{3 \times (4)^3}{12} + (12 \times 144) = 1744.$$

$$S = 12 \times 12 = 144.$$

Similarly for the triangle,

$$I = \frac{3 \times (4)^3}{36} + [6 \times (11\tfrac{1}{3})^2] = 771.$$

$$S = 6 \times 11\tfrac{1}{3} = 68.$$

Finally,
$$x_c = \frac{I}{S} = \frac{1744 + 771}{68 + 144} = 11.9 \text{ ft.}$$

The lateral position of the c. of p. may be determined from the fact that it must lie on the medial line. (See note in preceding para-

graph.) If y be its distance from the left hand edge of the figure, then

$$\frac{y}{3} = \frac{6.1}{8},$$

or $$y = 2.3 \text{ ft.} \quad Ans.$$

21. Relation between Center of Gravity and Center of Pressure.—The position of the c. of p. is always *below* the c. of g. of a surface. Were the intensity of pressure *uniform* over the surface, the resultant pressure would necessarily be applied at the c. of g. But since the pressures increase in magnitude as the head increases, the resultant pressure is carried downward and is always applied *below* the c. of g. If we pass from moderate to great depths, the variation in pressure on a surface becomes less and the c. of p. approaches the c. of g. as a limit. On small areas, therefore, located at depths of considerable magnitude, it is quite permissible to consider the positions of these two points as identical.

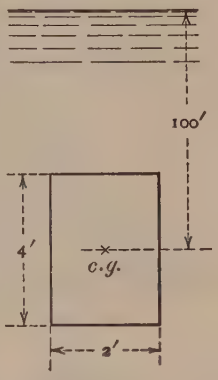


FIG. 16

Example.—Find the position of the c. of p. for the rectangle shown in Fig. 16.

From equation (11),

$$x_c - x_o = \frac{4 \times 4}{12 \times 100} = 0.013 \text{ ft.}$$

22. Component of Normal Pressure in a Given Direction.—Instead of finding the *total normal pressure* on a surface, it may be desired to find a component of this pressure parallel to some fixed direction. There arise several distinct cases.

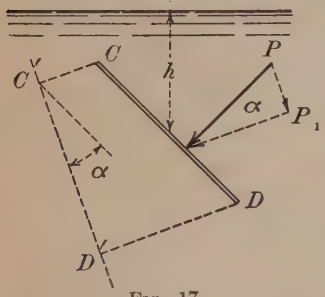


FIG. 17

Case (1). *Plane Surface.*—In Fig. 17, P_1 may represent the required component, making an angle α with the normal pressure P. The other component of P is, of course, perpendicular to the required component.

For the value of P_1 we have

$$P_1 = P \cdot \cos \alpha = A \, wh_0 \cos \alpha. \qquad (a)$$

The area A is represented by C-D in edge view and if C'-D' be the projection of A on a plane perpendicular to P_1, we have as a value of this projected area,

$$A_1 = A \cos \alpha,$$

from which and (a),

$$P_1 = A_1 w h_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

or P_1 is the product of the projected area and the intensity of pressure at the c. of g. of the original area. Hence the following

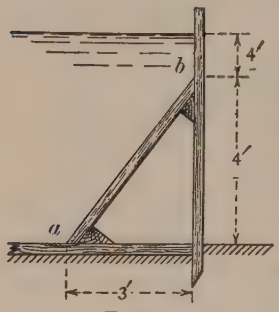


FIG. 18

Theorem.—To find the component of pressure parallel to a given direction, project the area upon a plane perpendicular to the component and multiply the projection by the intensity of pressure on the c. of g. of the original area.

Example.—Let it be desired to find the horizontal and vertical components of total normal pressure against the surface a-b, shown in Fig. 18, assuming that the length of a-b measured perpendicular to the plane of the paper is 10 ft.

By the above theorem,—

$$P_v = (3 \times 10) \times 62.5 \times 6 = 11250 \text{ lb.}$$

$$P_h = (4 \times 10) \times 62.5 \times 6 = 15000 \text{ lb.}$$

Case (2). *Curved and Irregular Surfaces.* (Fig. 19).—The following figure and demonstration are given for the special case of a surface of single curvature, although the same reasoning and results would be obtained from the consideration of an irregular or double curved surface. Again, C-D will represent the edge view of our surface. We may imagine it to be such a surface as might be formed from a bent iron sheet. If we consider a small elementary area dA, the normal pressure upon it is

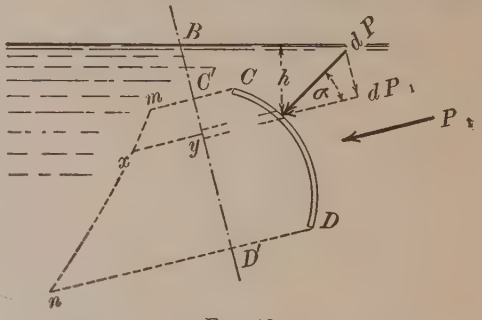


FIG. 19

$dP = dA \,.\, wh$, and $dP_1 = dP \,.\, \cos \alpha = dA wh \,.\, cos\ \alpha$.

Evidently the total value of P_1 may be expressed as

$$P_1 = \int dA \,.\, wh \cos \alpha, \qquad . \quad . \quad . \quad . \quad (15)$$

where h and $cos\ a$ are both variable, having, *in general*, no relation. The expression cannot then be simplified, but a graphical representation of P_1 may be made. If on a plane perpendicular to P_1 the area C-D be projected, we obtain an area representing the integral of $dA \,.\, \cos\ a$. On each projection of a dA we may imagine a normal ordinate to be erected equal to the h for the corresponding dA in the real surface. Such an ordinate is x-y. A little thought will show that they do not increase in length in proportion to their distance from B (measured on C'-D') and, therefore, m-n is not a straight line. Between the plane C'-D' and the curved surface m-n (in which lie the ends of all the ordinates) is a volume representing $\int h\, dA \cos \alpha$, and if this be considered a solid, having a unit weight w equal to that of water, then its total weight will be $P_1 = \int wh\, dA \cos \alpha$.

Reference to Fig. 19 will be made later in showing that in a special case the value of P_1 may be determined. Otherwise the figure has no practical application.

(3) **Special Cases.**—(*a*) *A curved or irregular surface with h assumed constant.* If in formula (15) the value of h be assumed constant, we have

$$P_1 = wh \int dA \,.\, \cos \alpha = A_1 wh,$$

which is the same as formula (14) for plane surfaces. Often in practical work the head is so large that the above assumption can be made.

(*b*) Again, if the conditions are such that the angle α is constant, we obtain from (15)

$$P_1 = w \cos \alpha \int dA \,.\, h = w \cos \alpha\, A h_0$$

$$P_1 = A_1 w h_0 \text{ as before.}$$

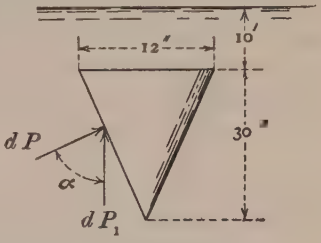


FIG. 20

Example.—As an illustration of this latter case, let it be desired to find the upward vertical component of normal pressure on the submerged cone shown in Fig. 20. It is quite obvious that the angle α is constant for all portions of the elementary

area, and we may proceed to find P_1 by projecting the conical surface upon a plane perpendicular to the direction of P_1 (*i. e.* a horizontal plane). The area thus obtained would be equal to that of the cone's base, or

$$A_1 = .7854 \text{ sq. ft.}$$
$$\therefore P_1 = A_1 w h_0 = .7854 \times 62.5 \times 10\tfrac{1}{2}\tfrac{0}{2}$$
$$\text{or } P_1 = 531.8 \text{ lb.} \quad Ans.$$

It should be noted that h_0 is the head on the center of gravity of the *conical surface* and not the c. of g. of the cone's surface as *a whole*. For the former, the c. of g. is located $\frac{2}{3}$ of the distance from the vertex to the base.

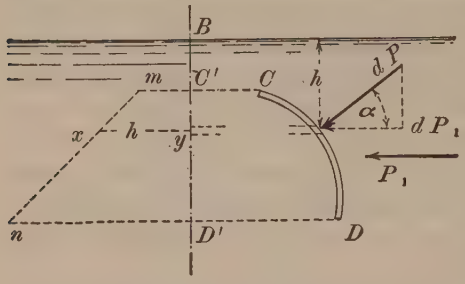


Fig. 21

(*c*) *Curved or Irregular Surface. Component horizontal.* — Perhaps most important of all to the engineer is the component of pressure in a horizontal direction; and it will now be proved that this may be found for *any kind* of surface. Referring to Fig. 21 and comparing it with Fig. 19, it will be seen that for this particular case each ordinate xy increases in direct proportion to its distance below the water surface, and the mean ordinate h_m will therefore be found at the center of gravity of the projected area, C'-D'. The volume formed from this area by the erection of ordinates on each dA will be $V = A_1 h_m$; and its weight, if of water, would be $W = A_1 w h_m$. It was pointed out, however, in Case 2 that this weight was the equivalent of P_1, so that we may write

$$P_h = A_1 w h_m, \quad . \quad . \quad (16)$$

or,

The horizontal component of pressure on an area is the same as the normal pressure on the vertical projection of the area.

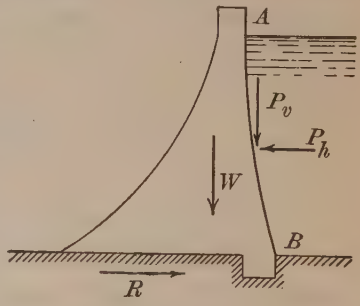


Fig. 22

Example (Fig. 22).—Let it be required to find the magnitude of the resistance R which the bed of the stream must be able to

offer the masonry dam in order to prevent its horizontal displacement by the pressure of the water. (Water 100 ft. deep.)

The forces acting on the dam are three in number; viz. its own weight W, the water pressure on A-B, and the reaction of the bed on the dam. (Of the latter force, R is the horizontal component.) Since these forces are in equilibrium, we may equate the algebraic sum of their horizontal components to zero, obtaining

$$P_h = R.$$

From (16)

$P_h = (100 \times 1)\ 62.5 \times 50 = 312500$ lb. per linear foot of dam.

(*d*) *Curved or Irregular Surface, Component Vertical.*—Another special case but not susceptible to treatment by either equation (15) or (16) is that of computing the *vertical* component of pressure on a curved or irregular surface. If we consider the prism of water vertically over such a surface, extending from it up to the free water surface, it may be seen, since the prism is in equilibrium, that the total upward supporting force on its base must equal its weight. The former must be equal to the vertical component of normal pressure on the given surface. The only difficulty arising in the computation of its numerical value is the determination of the volume of the prism. For some surfaces the solution is easy.

Note. The general and special cases treated in this article should be mastered by the student in order to avoid confusion of methods in the solution of practical problems.

23. Pipes and Cylindrical Shells under Pressure.—Water tanks, pipes, boiler shells and numerous structures designed or used by the engineer are commonly cylindrical in shape. When filled with water or other fluids under pressure, the shell is subjected to a stress which is commonly spoken of as "hoop tension." An expression for the value of its intensity may be found as follows.

Case I. *Shell with Axis Vertical.* (Fig. 23*b*).—Let us consider an elementary ring, A-B, of the shell which is subjected to a fluid pressure of intensity p. A horizontal section through the ring (Fig. 23*a*) shows this normal pressure, also one-half of the ring removed and its action on the remaining half indicated by the tensile forces, T, which are normal to CD and equal. Evi-

dently all the forces shown are in equilibrium and placing their X-components equal to zero, there results

$$2T = pd(\delta l)$$

$$T = \frac{pd}{2}(\delta l)$$

If the thickness of the ring be t, the average intensity of the hoop tension will be

$$f = \frac{T}{t(\delta l)} = \frac{pd}{2t} = \frac{pr}{t}. \quad . \quad . \quad . \quad . \quad (17)$$

For cylinders having thin walls, or walls whose thickness is small compared with the diameter, it is practically true that the intensity of stress is uniform throughout the wall.

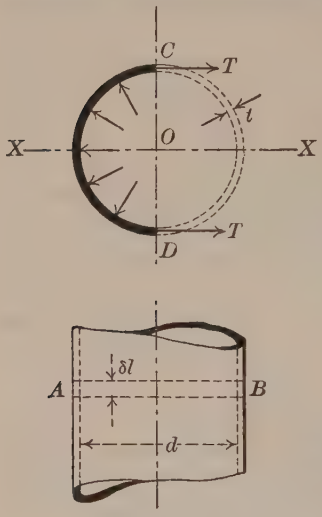


Fig. 23 A & B

For cylinders with thick walls the intensity probably is a maximum at the inner edge and a minimum at the outer. (For one treatment of this case see Gibson's Hydraulics, p. 39.)

The above equation may be written

$$t = \frac{pr}{f}. \quad . \quad . \quad . \quad . \quad (18)$$

giving the necessary thickness in the wall for any given intensity of fluid pressure.

Equations (17) and (18) are of importance to the designing engineer.

Case II. *Shells with Axis Horizontal.*—Water pipes, many tanks and boilers represent this case and the slight difference in stress, arising from placing the cylinder on its side instead of in a vertical position, should be noted. Figure 23 may be used for reference, but it must be noted (*a*) that the intensity of normal fluid pressure varies from the top to bottom of the shell, the average being found at the center; (*b*) that due to the weight of the enclosed water the shell tends to deform and assume a shape somewhat elliptical. This latter is prevented to a large extent by

the ability of the material in the pipe wall to offer resistance to bending. No exact analysis of the internal stress in the wall can, therefore, be made unless the stress arising from the bending be considered. In large shells this stress may be considerable in amount. Writers on the subject commonly neglect the bending stress and discuss the problem as though the cylinder was in a vertical position. With the cylinder horizontal and neglecting bending, Fig. 23*a* shows that we still have two tensile forces, T_1 and T_2, and a system of radial forces of varying intensity.

By Art. 22 the component of these latter, parallel to $X - X$, is $p_o d(\delta l)$ and

$$T_1 + T_2 = p_o d(\delta l)$$

As to the relation between T_1 and T_2, they must be equal, since the system of radial pressure may be replaced by a single resultant, acting through 0, having two components ΣH and ΣV. Taking moments about 0, of all forces acting on the half-ring, we find $T_1 = T_2$.

Therefore $$T = \frac{p_o d(\delta l)}{2},$$

and for the intensity we find

$$f = \frac{p_o r}{t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Also, $$t = \frac{p_o r}{f}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where p_0 represents the intensity of fluid pressure at the center of the shell.

If a study be made into the value of T, at other parts of the circumference, it will be found that (making the assumption of no bending), it is less in amount than at the top and bottom of the shell. Pipes and other horizontal shells should be given a greater thickness than figured from (20) to allow for stresses produced by bending, rough handling and uneven bearing.

24. Empirical Formula for Thickness of Pipe Walls.—Several formulas have been proposed for determining the thickness of pipes, all of them being empirical and designed to make allowance for the action of other forces than static water pressure. The best known of these is that proposed and used by the Metropolitan

Water Board of Boston. It has also been adopted by the New England Waterworks Association. In its usual form it is

$$t = \frac{(p + p')r}{3300} + 0.25$$

t = thickness in inches;
p = static pressure in pounds per sq. inch;
p' = allowance for water hammer in pounds per sq. inch;
r = radius of pipe in inches;
0.25 = allowance for eccentricity of shape, deterioration and safety in handling.

The value of p' is taken as follows.—

For diameter	3 to 10 inches,	$p' = 120$
	12 "	110
	16 "	100
	20 "	90
	24 "	85
	30 "	80
	36 "	75
	42 to 60 "	70

The formula assumes a strength of 16500 pounds per square inch and a factor of safety of 5.

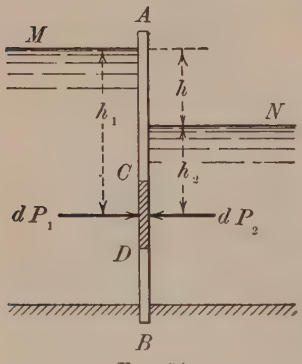


FIG. 24

25. Pressure on Both Sides of a Plane.—In the cases of immersed planes previously considered, the total pressure on one side equaled that upon the other, as the head on the c. of g. was unchanged. Imagine, however, a plane such as A-B (Fig. 24) separating the two water levels M and N, and let C-D be an edge view of a certain portion of its area. The total pressure on any small dA will be, for the left-hand side,

$$dP_1 = dA \,.\, wh_1,$$

and for the right-hand side,

$$dP_2 = dA \,.\, wh_2$$

Their *resultant* (acting to the right) will be

$$dR = dP_1 - dP_2 = dAw(h_1 - h_2) = dA \,.\, wh.$$

That is, the resultant pressure on any elementary area is dependent only on h, the difference in water levels. Each dA of C-D then has the same resultant pressure, and a single resultant applied at the c. of g. of C-D may replace these equal pressures. This is graphically shown in Fig. 25. The variation of pressure over A-B may be represented, on the left side, by the straight line O-E, and on the right side by A-F. The triangles O-B-E and A-B-F are similar, and A-B-F and C-D-E are equal. The trapezoid A-C-B-E represents the total pressure on the left of A-B, from which we must subtract the triangle A-B-F, or its equal, C-D-E, in order to obtain the resultant pressure. This will give the rectangle of pressure A-B-C-D, which shows the resultant pressure to be uniformly distributed over A-B.

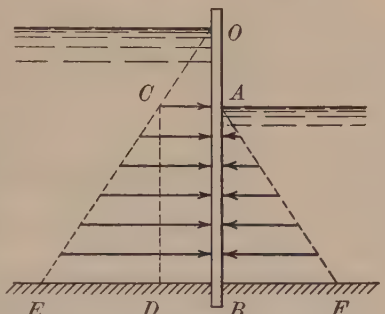


FIG. 25

26. Pressure on Immersed Bodies.—*Theorem of Archimedes.* About the year 250 B. C. Archimedes discovered the important law that immersed bodies lose a portion of their weight equal to that of the volume of water displaced by them. The proof is simple (Fig. 26). AB is any such body through which we will pass a vertical plane C-D (perpendicular to plane of paper). The horizontal components of total pressure H_1 and H_2, on the irregular areas mAn and mBn, must be equal, as both are measured by the same projected area m-n (Art. 22). If the vertical plane C-D were passed through the body parallel to the plane of the paper, we should obtain $H_3 = H_4$, these latter forces being at

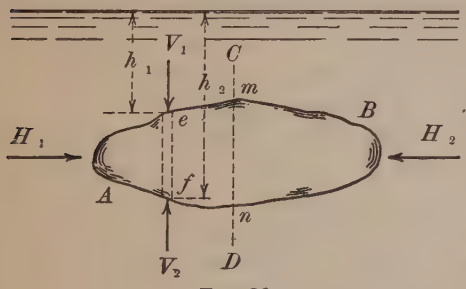


FIG. 26

right angles to H_1 and H_2. Evidently $\Sigma H = 0$ for all the normal pressures. To investigate the vertical components, assume a vertical prism *e-f* with end areas so small as to give uniform intensity of pressure upon them. By Art. 22, under these conditions we have

$$V_1 = dA \,.\, wh_1,$$

dA being the area of the prism's cross-section. Similarly we obtain

$$V_2 = dA \,.\, wh_2;$$

and the net resulting vertical force on the prism is

$$dR = V_2 - V_1 = dA \,.\, w\,(h_2 - h_1).$$

This latter term is the weight of a volume of water equal to that of the prism. By a consideration of every elementary prism in the body, we may conclude that the body as a whole is acted upon by an upward resultant force equal to the weight of the volume of water displaced by the body.

If the body be floating at the surface with only a portion immersed, the law still holds good, as a consideration of the vertical pressures on an elementary prism will show. In either case we may conceive of two forces acting on the body,—the weight of the body and the *buoyant effort* of the water. The weight acts through the center of gravity of the body, while the buoyant effort acts through the center of gravity of the volume of displaced liquid. This may be seen by a return to an elementary prism. The small upward resultant force acting upon it being *proportional to its volume*, it follows that the final resultant of these elementary resultants must act through the center of gravity of the total volume. This point is called the *Center of Buoyancy*.

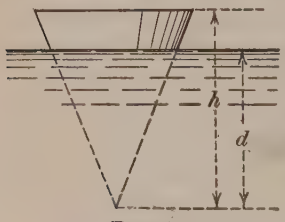


FIG. 27

27. Depth of Flotation.—To find the depth to which a floating body will sink, it is only necessary to remember that the weight of the body equals the weight of the displaced water. From the geometrical relations for any regularly shaped body we may then figure the distance.

Illustration.—Pyramid or cone with axis vertical and vertex down (Fig. 27).

Let V = volume of cone.
w' = weight of cone per cubic unit.
v' = volume of immersed part.

$$\text{Then } \frac{v'}{V} = \frac{d^3}{h^3}, \text{ and } v' = \frac{d^3}{h^3} V.$$

The weight of the cone is $w'V$, and the weight of the water displaced is $w\frac{d^3}{h^3}V$, hence

$$w'V = w\frac{d^3}{h^3}V \text{ and } d = h\sqrt[3]{\frac{w'}{w}}.$$

The ratio $\frac{w'}{w}$ equals s, the specific gravity of the solid, so that finally,

$$d = h\sqrt[3]{s}.$$

28. Stability of Immersed and Floating Bodies.—If the weight of a body exceeds the buoyant effort, it will sink. If it be less, it will, if placed beneath the surface, rise to the surface and assume a position according to the above stated principles. If the weight were just equal to the buoyant effort, it would remain beneath the surface wherever placed; or, if placed upon the surface, would sink until just submerged. Reference to Fig. 28 will show that for this latter case the body will assume a position so that its center of gravity and the center of buoyancy will lie in the same vertical line. This is necessary for equilibrium, as otherwise the weight at the center of gravity and the upward force at the center of buoyancy form a couple which will rotate the body. If the center of gravity is below the center of buoyancy, the couple will right the body and the equilibrium is stable. If the center of gravity be *above*, the condition is clearly that of unstable equilibrium; and if the two points coincide, the body is in equilibrium for all positions.

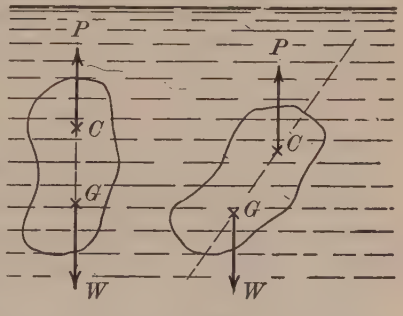


FIG. 28

If the body *floats*, the conditions of equilibrium are not so easily determined. The position of the center of gravity will of course

remain unchanged, but the volume of displaced water will alter its shape as the body is rotated, involving a change in the position of the center of buoyancy. This is well shown in the case of a floating ship (Fig. 29). Assuming the center of gravity to be at G and the center of buoyancy at C, or below G, it would seem at first sight as though the ship were in unstable equilibrium. However, consider an outside force to be applied causing the ship to roll (Fig. 30). The position of C is changed to some point C' lying on the side of greater immersion. In the figure shown, the line of action of the upward force cuts the axis A-B at a point M *above* the center of gravity. Evidently for this particular case the equilibrium is stable, as the couple tends to right the ship. If M had fallen *below* G, the action of the couple would have tended

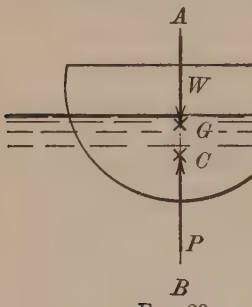


Fig. 29

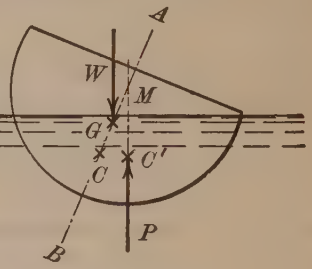


Fig. 30

to upset the ship, and the equilibrium would have been unstable. The point M is called the *Metacenter*. Its position relative to the center of gravity becomes then a criterion of stability, and it is important that we have some means for determining its exact location. As a matter of fact its position changes with different values of θ, the angle of displacement. As this angle becomes smaller and smaller, approaching zero as its limit, M approaches a limiting point which is the *true* metacenter, the position of M when θ = zero. It can be quite easily proved that the following equation will give the distance h_m measured from the center of gravity G (Fig. 30) to the metacenter M.

Let V = volume of displaced water.

I = least moment of inertia of a *water line* section through the body, about an axis through its own c. of g.

c = distance between c. of g. and c. of b.
= CG (Fig. 29).

Then $$h_m = \frac{I}{V} \pm c. \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Here c is minus when the c. of g. is above c. of b.

For the derivation of this formula and a more detailed treatment of the subject the student is referred to an excellent article by Professor I. P. Church in his "Mechanics of Engineering," John Wiley & Sons, New York, also by A. H. Gibson, in "Hydraulics and Its Applications," Van Nostrand, New York.

PROBLEMS

1. A gas holder contains illuminating gas under a pressure corresponding to 2 in. of water. If the holder be at sea level (atmospheric pressure 14.7 lb. per square inch), what pressure in inches of water may be expected in one of its distributing pipes at a point 500 ft. above sea level? Assume unit weights of air and gas to be constant at all elevations with values of 0.08 and 0.04 lb. per cubic foot respectively.

Ans. 5.84 in.

2. A rectangular plate is immersed vertically with one of its sides in the water surface. How must a straight line be drawn from one end of that side so as to divide the rectangle into two areas, the total pressures upon which shall be equal?

3. Two vessels, A and B, containing water under pressure, are connected by an oil differential gauge of the type illustrated in Fig. 9. If a point m in A is 4.85 feet below a point n in B, find the difference in pressure at these points when the top of the water column in the tube entering A stands 15 inches below that in the tube entering B. Specific gravity of oil is 0.80.

Ans. 2.0 lb. per square inch.

4. A differential gauge is used to measure the air pressure in a ship's stokehold. It is in the form of a vertical, glass U-tube, partly filled with water, having one leg in the stokehold and the other outside where the pressure is atmospheric. Oil with a specific gravity of 0.80 fills the inside leg, the line of separation between oil and water being in this leg. The upper ends of the tube are enlarged so that the sectional area is 10 times that of the rest of the tube, and the top surfaces of the oil and water are in the enlarged portions. Find what increase in pressure (over that of the atmosphere outside) in the stokehold will force the line of separation downward 1 inch.

Ans. $p = 0.014$ lb. per square inch.

5. A liquid whose specific gravity is 1.25 partly fills the vessel shown in Fig. 6. What will be the intensity of pressure at a point 1.8 feet below its surface if h be 15 inches of mercury?

6. Water fills the two vessels shown in Fig. 10, and a portion of the connecting tube. If the oil has a specific gravity of 0.85, what will be the difference in pressure-intensity at the points m and n when $h = 4$ ft. and $z = 10$ inches?

7. If mercury, instead of water, occupies the vessels and portions of the tube shown in Fig. 10, what will be the difference in pressure-intensity at points m and n when $h = 12$ inches, $z = 18$ inches, and the oil has a specific gravity of 2.0?

8. A masonry dam, 70 ft. high, has a triangular cross-section, the down-stream side being vertical. The masonry weighs 150 lb. per cu. ft. and the water back of the dam has a depth of 60 ft. above the latter's base. What width must the dam have at its base in order that the resultant of the weight of the dam and the pressure of the water shall fall within the base at a distance from the down-stream face equal to $\frac{4}{10}$ the base width?

Ans. 40 ft.

9. Compute the total normal pressure on the interior sides and bottom of a cylindrical water tower, 40 ft. in diameter, containing water to a depth of 60 ft.

Ans. 18,800,000 lb.

10. A tank with plane vertical sides contains 4 ft. of mercury and 12 ft. of water. Find the total pressure on a portion of a side, 1 ft. square, one half this portion being below the surface of the mercury. The sides of the square area are vertical and horizontal.

Ans. 849 lb.

11. A vertical, rectangular sluice gate at the bottom of a dam is 2 ft. wide, 6 ft. high, and exposed to water pressure on one side corresponding to a head of 50 ft. above its center. Assuming the gate and stem to weigh 500 lb. and the coefficient of friction of gate on runners to be 0.25, find the force necessary to raise it.

Ans. 9875 lb.

12. A vertical gate, 4 ft. wide and 6 ft. high, hinged at the upper edge, is kept closed by the pressure of water standing 8 ft. deep over its top edge. What force applied normally at the bottom of the gate would be required to open it?

Ans. 9000 lb.

13. How far below the water surface is it necessary to immerse a vertical plane surface, 3 feet square, two edges of square being horizontal, in order that center of pressure shall be but 1 inch from center of gravity?

14. An immersed plane surface has the form of a square, 6 ft. on a side, but from an upper corner (plane vertical and sides of square horizontal and vertical) a piece has been taken by a straight cut leaving the top edge 4 ft. long and the vertical edge 2 ft. long. Locate the center of pressure both vertically and laterally.

15. A vertical triangular plate whose height is 12 ft. has its base horizontal and vertex uppermost in the water's surface. Find the depth to which it must be lowered so that the difference in level between the center of gravity and the center of pressure shall be 8 in.

Ans. 4 ft.

16. Find the depth to the center of pressure on a trapezoidal surface, vertically immersed in water, the upper base being 5 ft. long, parallel to and 10 ft. below the water surface. The trapezoid is symmetrical about a vertical center line, its lower base being 3 ft. long and 13 ft. below the surface.

Ans. 11.44 ft.

17. A flat parabolic plate is immersed, with its axis vertical; in water until its vertex is 7 ft. below the water surface. Locate the center of pressure.

Ans. 4.0 ft. down.

18. Compute the position of the center of pressure on a circular gate, 4 ft. in diameter, placed with its center 4 ft. below the water surface and in a plane inclined 45 degrees from the vertical.

Ans. 0.18 ft. below center of gravity.

19. A plate shaped as a right triangle is immersed in water with one side vertical. If the head on the upper vertex be 3 ft., the length of the vertical side, 6 ft., and that of the horizontal side, 10 ft., locate the center of pressure. Find its lateral position by the calculus method and check by means of medial line.

Ans. 0.29 ft. below center of gravity.
3.57 ft. from vertical side.

20. An isosceles triangle, base 10 ft. and altitude 20 ft., is immersed vertically in water with its axis of symmetry horizontal. If the head on its axis be 30 ft., locate the center of pressure both laterally and vertically.

Ans. $x_c = 30.14$ ft.
$y_c = 6.67$ ft. from base.

21. A triangle having a base of 4 ft. and an altitude of 6 ft. is wholly immersed in water, its base being in the surface and its plane vertical. Find the ratio between the pressures on the two areas into which the triangle would be divided by a horizontal line through its center of pressure.

22. A vertical, rectangular sluice gate, 12 ft. high and 10 ft. wide, is hinged so as to revolve about a horizontal axis placed 4 inches below its center of gravity. How deep will the water have to stand on the top edge of the gate if it is to be in equilibrium under the pressure of the water?

23. A vertical parabolic plate is immersed vertex down in water. If the head on the vertex be 9 ft., and the width of plate at the water line be 4 ft., compute the total pressure on one side of the plate and the position of the center of pressure.

24. Compute the stress in a 36-inch pipe, whose walls are $\frac{3}{8}$ in. thick, if the water which fills it is under a pressure equivalent to 230 ft. of head on its center.

Ans. 4800 lb. per square inch.

25. A wood stave pipe, 48 in. in inside diameter, is to resist a maximum water pressure of 150 lb. per square inch. If the staves are bound together by flat steel bands, 4 in. wide by $\frac{3}{4}$ in. thick, find the spacing distance of the latter in order that they may not be stressed above 15,000 lb. per square inch.

Ans. 12.5 in.

26. If a 12-inch flanged pipe be closed at its end by a hemispherical cap bolted to the flange, what total stress will be in the bolts when the head on the pipe's center is 240 ft.?

Ans. 11,800 lb.

27. What thickness should be given the steel wall of a 60-inch pipe if it is to withstand a pressure of 100 lb. per square inch with a maximum fiber stress of 15,000 lb. per square inch?

Ans. 0.2 in.

28. A timber dam has a plane up-stream face sloping at an angle of 60 degrees with the horizontal. Compute the vertical and horizontal components of the pressure (per linear foot) against it when water stands 30 ft. deep behind the dam. Compare these figures with those obtained for a slope of 45 degrees.

Ans. (*a*) V = 16,250,
H = 28,150.
(*b*) V = 28,150,
H = 28,150.

29. The flat bottom of a steel tank is connected with the plane, vertical side by a plate curved through 90 degrees on a radius of 2 ft. If water stands to a depth of 8 ft. in the tank, compute the horizontal and vertical components of the normal pressure on a linear foot of the curved plate.

Ans. V = 946 lb., H = 875 lb.

30. A rectangular caisson is to be sunk, in which to build the foundation for a bridge pier. It is in the form of an open box, 50 ft. by 20 ft. in plan, and 23 ft. deep. If it weighs 75 tons, how deep will it sink when launched? The water being 20 ft. deep, what additional load will sink it to the bottom?

Ans. 2.4 ft.
550 tons.

31. A flat-bottomed scow is built with vertical sides and straight sloping ends. Its length on deck is 80 ft., on the bottom 65 ft., its width 20 ft., and its vertical depth is 12 ft. How much water will it draw if it weighs 250 tons?

Ans. 5.8 ft.

32. A ship with cargo weighs 5000 tons and draws 25 ft. of water. On crossing a bar at the entrance to a river her draft is decreased by 1 ft. by the discharge of 300 tons of water ballast. In going up the river to fresh water, 50 tons of coal are burned. What will her draft be then and how much ballast will be required to increase it by 1 ft.? Assume the unit weight of salt water to be 64.0 lb. (SUGGESTION: The shape of the vessel's under body is not known and the problem should be solved on the basis of displacement in cubic feet. The sides of the vessel near the water line may be assumed vertical.)

Ans. 24.21 ft.
293 tons.

33. A hollow cylinder 3 ft. long and 3 ft. in diameter, closed at one end, is immersed with axis vertical and closed end uppermost, this end being held 25 ft. below the water surface. The cylinder is at first full of water, but compressed air is admitted from beneath in the immersed position until it has displaced two thirds of the water.

Find:

(*a*) Absolute pressure from above on top of cylinder.
(*b*) Absolute pressure from below on top of cylinder.
(*c*) Supporting capacity of cylinder due to buoyancy.
(*d*) Supporting capacity of cylinder if it be allowed to rise 15 ft.
(*e*) Position of maximum supporting capacity with same charge of air.

Ans. (*a*) 26,040 lb. (*b*) 26,940 lb. (*c*) 900 lb.
(*d*) 1155 lb. (*e*) 3.6 ft. head on top.

34. A tank with vertical sides is 4 ft. square, 10 ft. deep, and is filled to a depth of 9 ft. with water. By how much, if at all, will the pressure on one side of the tank be changed if a cube of wood, specific gravity 0.5, measuring 2 ft. on an edge, be placed in the water so as to float with one face horizontal?

Ans. 575 lb.

35. A cask which weighed 60 lb. was placed on platform scales and then nearly filled with water. A total load on the scales of 320.25 lb. was read. Should the net weight of water as computed from these figures be corrected by reason of the fact that a 3-inch vertical steel shaft suspended from the ceiling above had its lower end immersed in the water to a depth of 1 ft.? If so, by what amount?

Ans. 3.06 lb.

36. If the specific gravity of a body is 0.8, what proportional part of its total volume will be above the surface of the water upon which it floats?

Ans. 0.20.

37. A stick of yellow pine timber, weighing 40 lb. per cubic foot, measures $6'' \times 12'' \times 20'$. What load will it carry without sinking, if placed in fresh water?

Ans. 225 lb.

38. A vertical cylindrical tank, open at the top, contains 12,000 gals. of water. It has a horizontal, sectional area of 80 sq. ft. and its sides are 40 ft. high. Into it is lowered another similar steel tank, having a cross-section of 60 sq. ft. and a height of 40 ft. The second tank is inverted so that its open end is down, and it is allowed to rest on the bottom of the first tank. Find the maximum hoop tension per vertical inch in the outer tank. Neglect the thickness of the metal forming the inner tank.

39. A ship with a total displacement of 1800 short tons rolls to one side through an angle of 1 degree when a deck load of 5 tons is moved laterally through a distance of 15 ft. Compute the metacentric height for this particular position of the vessel.

Ans. 2.39 ft.

40. A rectangular pontoon is 80 ft. long, 40 ft. wide, and 12 ft. deep. Its draft when launched is 4 ft. and is increased to 10 ft. when fully loaded. Compute the position of the true metacenter for both drafts. Assume in each case the center of gravity of the pontoon and load at the geometrical center of the cross-section.

Ans. 29.3 ft., light.
12.4 ft., loaded.

CHAPTER III

EFFECTS OF TRANSLATION AND ROTATION UPON BODIES OF WATER

29. Body of Water in Straight Line Motion.—(*a*) *Direction Horizontal.* Under this heading will be presented the single case of a mass of water having a motion of pure translation, but with its individual particles in a state of relative rest. Such a case may be illustrated by Fig. 31, in which is shown a reservoir partially filled with water and moving horizontally to the right. If the motion be *uniform*, then the forces acting on any particle are balanced and conditions are identical with those existing when the body is at rest.

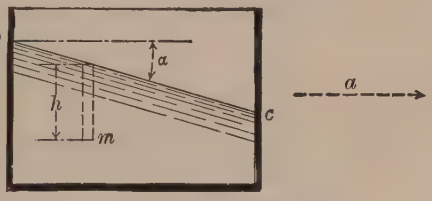


FIG. 31

That is to say, the free surface will be horizontal and the pressure at any point in the mass follows the static laws already explained. Let us consider, however, that the reservoir has a uniformly *accelerated* motion toward the right, a being the value of the acceleration. If the reservoir starts from a state of rest, the surface bc will be observed to at first oscillate and then come to rest, occupying some such position as shown in the figure, making an angle α with the horizontal which we may determine as follows. Any particle in the surface must experience a single resultant force F, acting horizontally to the right, since its motion is uniformly accelerated and in that direction *only*. The measure of this force we know to be

$$F = Ma.$$

Since upon each particle there is acting the force of gravity W, there must be present another force P which, when combined with W, gives F as a resultant. The parallelogram of forces

(Fig. 32) shows that P must act obliquely upward. It represents a force exerted by the surrounding particles, and if the particle considered is to remain at rest relative to its neighbors, this force must act normal to the free surface. The angle between P and the vertical is the angle α which the surface bc makes with the horizontal. The value of F as obtained from the figure is

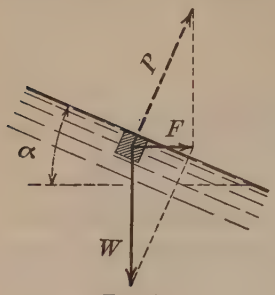


FIG. 32

$$F = W \tan \alpha,$$

and already we have

$$F = Ma = \frac{W}{g}a.$$

Eliminating W and F, we finally obtain

$$\tan \alpha = \frac{a}{g} \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

A little thought will show that this same result would follow, and bc occupy the same position, had the reservoir been moving to the *left* with motion uniformly *retarded.*

At any depth, h, measured vertically downward from a point in the surface, the intensity of pressure will be wh as may be seen from a consideration of all the forces acting on an imaginary elementary prism of water extending from any point, m, (Fig. 31) up to the free surface. Since the prism has no vertical acceleration, it follows that ΣV must be equal to zero. The vertical forces are the weight of the prism, $wh(dA)$, and the upward force $p_m(dA)$ on its base.

Therefore $$p_m(dA) = wh(dA)$$

and $$p_m = wh.$$

(*b*) *Direction Vertical.* If the reservoir be moving in a vertical path (downward or upward) with *uniform* velocity, the surface bc will remain horizontal, and the pressure throughout the water will be the same as if it were at rest. If the motion be uniformly *accelerated*, the surface will still remain horizontal, but the pressures in all parts of the water will change. Let the direction of motion be upward and the acceleration be positive in the same direction. Any elementary prism, as mn, we may treat as a

rigid body in motion under the action of the forces shown (Fig. 33). Its weight is $dA \,.\, w \,.\, h$, and on its base is the total pressure $p \,.\, dA$. The horizontal forces (side pressures) are omitted, as they produce no vertical motion. The resultant force F equals mass $\times$ acceleration, or

$$p \,.\, dA - dA \,.\, w \,.\, h = \frac{dA \,.\, w \,.\, h}{g} \times a,$$

from which

$$p = wh\left(\frac{g + a}{g}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

For the special case of

$$a = g,$$

we have

$$p = 2\ wh.$$

If the acceleration be negative (*i. e.* mass moves upward with decreasing velocity), then a in (23) becomes negative. Thus, a mass of water moving upward, but coming to rest under the action of gravity alone, will have as a pressure in all parts,

$$p = wh\left(\frac{g - g}{g}\right) = 0.$$

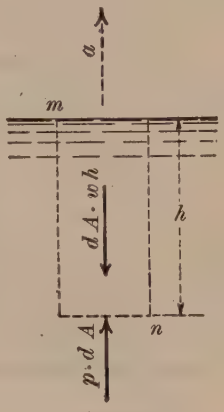


FIG. 33

If the direction of motion now be changed to *downward*, with the acceleration positive in the *same direction*, a reconsideration of the forces acting on the elementary prism will show that

$$p = wh\left(\frac{g - a}{g}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

If the body of water falls freely,

$$p = wh\left(\frac{g - g}{g}\right) = 0.$$

If the acceleration be in a direction opposed to motion, the negative sign in (24) becomes positive.

A little thought will make it clear that the law of pressure for a body moving downward with increasing velocity is the same as though moving upward with decreasing velocity; and for a body

moving downward with decreasing velocity, the same as though moving upward with velocity increasing.

It has just been shown that, in the case of water *freely falling*, the pressure at any point in the mass is zero. This principle finds a very important application in the case of a *stream* of water falling through space. Save for the resistance of the atmosphere, all particles of the stream are *freely falling* and hence there is no mutual pressure exerted by contiguous particles. Streams issuing from pipes, nozzles, orifices, etc., are illustrative examples.

30. Body of Water in Rotation.—If an open vessel of any shape be partly filled with water and made to rotate at a fixed rate about any vertical axis passing through the mass of water, it will be found that, as the contained water acquires the angular velocity of the vessel, its free surface, at first horizontal, becomes dished or concave in form. (Fig. 34.) The mathematical nature of this surface and the reason for its formation may be understood if we consider the forces acting upon the fluid particles forming it. Selecting a small mass, located a distance x from the axis of rotation, let us designate its weight by W. The only other forces acting upon it are the pressures it receives from the surrounding particles, and since it has no motion relative to the latter, the direction of the resultant, F, of these pressures must be normal to the curved surface. The resultant of W and F must be the horizontal *deviating* force C, whose magnitude is $Mu^2 \div x$, u representing the linear velocity of the small mass. (The resultant, C, must be horizontal and act toward the center of rotation since the small mass is following a horizontal path and is being *uniformly accelerated toward the center of rotation.*) If ω (Omega) represents the angular velocity, so that $u = \omega x$,

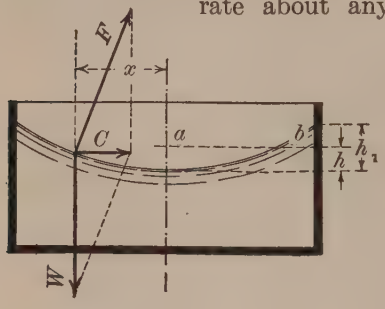


Fig. 34

$$C = \frac{M\omega^2 x^2}{x} = \frac{W}{g}\omega^2 x.$$

Referring to Fig. 35, the tangent of the angle θ which F makes with the vertical is

$$\tan \theta = \frac{C}{W} = \frac{\omega^2 x}{g}.$$

From the figure, θ is also the angle which a tangent to the curved surface, at the point under consideration, makes with the axis, X. The tangent of θ may, therefore, be written

$$\tan \theta = \frac{dy}{dx} = \frac{\omega^2 x}{g}$$

from which,

$$dy = \frac{\omega^2 x}{g}\, dx,$$

and

$$y = \frac{\omega^2 x^2}{2g}. \quad \ldots\ldots\ldots \quad (25)$$

The equation is that of a parabola with its vertex on the axis of rotation, and the form of the free surface is, therefore, that of a paraboloid of revolution. Such a surface will always be formed, when a mass of water is rotated about a vertical axis, provided the surface be *free*. The axis may or may not pass through the mass itself, but the vertex of the paraboloid formed will always be on the axis.

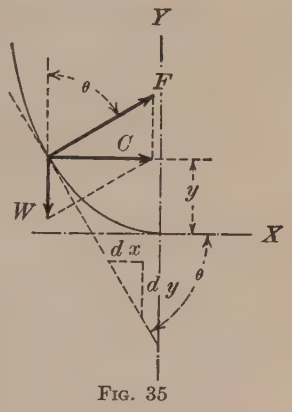


FIG. 35

Equation (25) may be more conveniently remembered, perhaps, if it be written

$$y = \frac{u^2}{2g}. \quad \ldots\ldots\ldots \quad (26)$$

u being the *linear* velocity of any point on the surface, and y its elevation above the vertex.

At any depth, h, measured vertically downward from a point in the surface, the pressure intensity will be wh as may be seen from a consideration of the forces acting on an imaginary elementary prism of water extending downward from the point a distance h. The prism having no *vertical* acceleration, it follows that ΣV must be equal to zero, and the upward pressure, $p(dA)$, on the base

of the elementary prism must equal the weight, $wh(dA)$, of the latter. Therefore $p = wh$.

31. Rotating Cylindrical Vessels with Free Surfaces.—If the containing vessel be cylindrical in form with the axis of symmetry also the axis of rotation, we may ascertain the relation in space of the new surface to the original surface. Thus assuming Fig. 34 to represent a section through such a vessel, the original level may be shown by the line a-b, and A will denote the horizontal cross-sectional area of the vessel. Remembering that the volume of a paraboloid is one half that of the circumscribing cylinder, we have

$$\text{Vol. paraboloid} = Ah_1 \div 2$$

while from the figure we see that

$$A\,h_1 \div 2 = A\,h$$

and therefore

$$h = \frac{h_1}{2}.$$

That is, the distance between the vertex of the paraboloid and the original water level is the same as from the original level to the highest point on the new surface. Evidently this fact holds for only the cylindrical vessel.

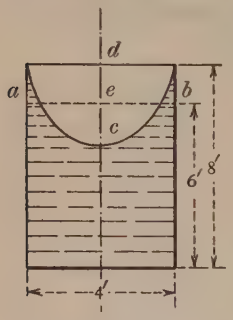


FIG. 36

Example.—A cylindrical vessel (Fig. 36) is filled with water to a depth of 6 feet. Its height being 8 ft., and radius 2 ft., determine the angular velocity and revolutions per minute which will raise the water even with the brim. Find also the total pressure on the sides of the vessel.

Solution.

$$ce = cd = 2 \text{ ft.}$$

$$\therefore cd = 4 \text{ ft.} = \frac{\omega^2 x^2}{2g} = \frac{\omega^2 4}{64.4}$$

$$\omega = 8.03 \text{ radians per second.}$$

$$\omega = \frac{8.03}{2\pi} \times 60 = 76.6 \text{ R.P.M.},$$

since 2π radians correspond to one revolution.

To find the total pressure on the side:—

$$P = Awh_0 = (\pi \times 4 \times 8)\, 62.5 \times 4 = 25100 \text{ lb.} \quad Ans.$$

32. Rotating Vessels with Water Surface Confined.—If a vessel be entirely filled with water and closed at its top so as to confine the water, it will, of course, be impossible to form the parabolic surface by uniform rotation about a vertical axis. However, the intensity of pressure at any point in a horizontal plane through the mass will increase as the square of the distance from the axis, just as in the previous cases. This may be proved with the aid of Fig. 37, in which an imaginary elementary prism of water connects the point, m, with the axis of rotation. At either end of the prism acts a horizontal force produced by the adjacent water particles. There are no other horizontal forces acting, and that at m must exceed the other by an amount sufficient to produce the normal acceleration which the prism has as it rotates. If its rate of angular rotation be ω, the necessary difference in the two forces must be

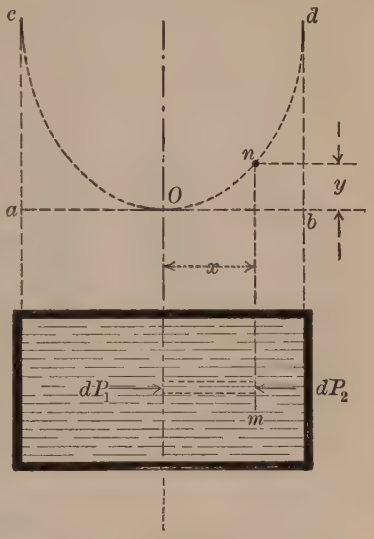


FIG. 37

$$dP_2 - dP_1 = dM.\omega^2 \frac{x}{2} = \frac{w.dA.x^2.\omega^2}{2g}.$$

Dividing by dA,

$$p_2 - p_1 = \frac{wx^2}{2g}\omega^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

showing that the pressure intensity varies as the square of the distance out from the axis.

Another form of (27) is

$$\frac{p_2}{w} - \frac{p_1}{w} = \frac{\omega^2 x^2}{2g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

giving the difference in *pressure-head* between the two points. A little thought will show that the pressure intensity at any point on the axis will not be changed by the rotation of the mass as the latter produces no motion in particles on the axis. Equation (28), therefore, gives the increase in head, produced by rotation, at any distance x from the axis. It is identical with equation (25).

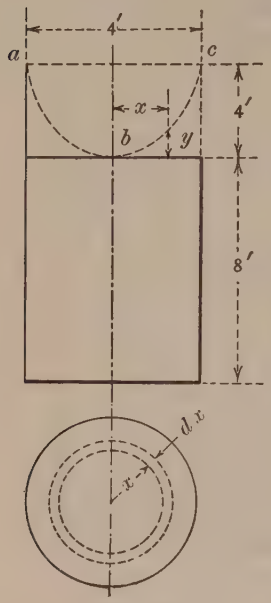


Fig. 38

A graphical representation of the above conditions may be obtained by drawing in Fig. 37 the parabolic arc, *c-o-d*, with its vertex on the axis and tangent to a line *a-b* which represents by its height above the confined water surface the pressure-head on the surface before rotation commenced. A vertical ordinate from any point, *m*, to this parabola measures the pressure-head at the point during rotation. The equation of the parabola from equation (28) must be

$$y = \frac{\omega^2 x^2}{2g};$$

if y measures the increase in pressure-head at a distance x from the axis.

Example.—If the cylindrical vessel shown in Fig. 38 be just filled with water and closed at the top, determine the total pressure on the top, bottom, and sides when rotated about its own axis at a rate of 76.6 R.P.M.

It will be seen that, during rotation, the variation in pressure-head over the top surface of the liquid is represented by the parabolic curve, *a b c*, whose maximum ordinate is

$$y = \frac{\omega^2 x^2}{2g} = \frac{8.03^2 \times 4}{64.4} = 4 \text{ ft.}$$

(76.6 R.P.M. = 8.03 radians per sec.)

Pressure on sides,—

$$P = Awh_0 = (\pi \times 4 \times 8)\, 62.5 \times 8$$
$$= 50200 \text{ lb.} \quad Ans.$$

Pressure on top,—
On an elementary ring (Fig. 38),

$$dP = dA\ wh = (2\pi x.dx)\,wy,$$

y having the value $\frac{\omega^2 x^2}{2g}$.

$$\therefore P = \frac{\pi\omega^2 w}{g}\int_0^2 x^3 dx = 1570 \text{ lb.} \quad Ans.$$

Pressure on the base,—

$$P = Awh + 1570$$

$$= (\pi 4 \times 62.5 \times 8) + 1570 = 7850 \text{ lb.} \quad Ans.$$

The discussion in the previous paragraph explains in a crude way the operation of a centrifugal pump. By the rotation of the mass of water contained in the *impeller* or *rotor* of the pump, the pressure-head is greatly increased in the outer peripheral space which receives the water from the impeller. The problem of analyzing the action of a centrifugal pump involves kinetics and does not come within the scope of the present chapter.

PROBLEMS

1. What distance must the sides of a tank be carried above the surface of water contained in it, if the tank (moving horizontally) is to suffer an acceleration of 10 ft. per second each second without losing water? Tank is 6 ft. square with water 3 ft. deep. Compute the maximum intensity of pressure on the bottom of the tank during acceleration.

Ans. (*a*) 0.93 ft.
(*b*) $p = 1.75$ lb. per square inch.

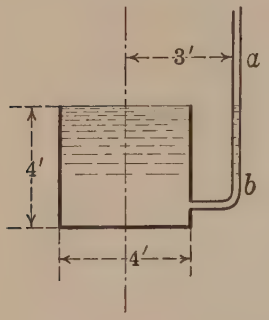
2. How much water will be spilled from a rectangular tank, 5 ft. long, 3 ft. wide and 4 ft. deep, if starting from a state of rest and full of water, it be horizontally accelerated in the direction of its length at a rate of 2 ft. per second each second?

3. An open tank, 30 ft. long, is supported on a car moving on a level track and uniformly accelerated from rest to 30 miles per hour. When at rest, the tank was filled with water to within 6 inches of its top. Find shortest time in which the acceleration may be accomplished without liquid spilling over the edge.

4. A vessel containing oil (specific gravity 0.70) moves in a vertical path with an acceleration of 8 ft. per second each second. Find the intensity of pressure at a point in the oil 3 ft. beneath its surface when,

(*a*) moving upward with positive acceleration.
(*b*) moving upward with negative acceleration.
(*c*) moving downward with positive acceleration.
(*d*) moving downward with negative acceleration.

5. If the water which just fills a hemispherical bowl of 3 ft. radius be made to rotate uniformly about the vertical axis of the bowl at the rate of 30 revolutions per minute, how much will overflow? *Ans.* 19.5 cu. ft.

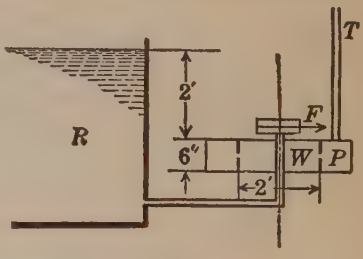


6. The open cylindrical vessel shown in the accompanying sketch is revolved about its center axis at the rate of 56 R.P.M. If previously filled with water to the brim, how high above the latter will water rise in the attached piezometer tube, *a-b*?

Ans. 2.68 ft.

7. At what speed must an open, vertical cylindrical vessel, 4 ft. in diameter, 6 ft. high, and filled with water, be rotated on its own axis in order that the effect of rotation will be to discharge sufficient quantity of water to uncover a circular area on the bottom of the vessel 2 ft. in diameter?

Ans. 108 R.P.M.



8. Water from the constant-level reservoir, *R*, flows to, and entirely fills, the closed, concentric cylindrical chambers shown in the figure. A wheel, *W*, composed of flat vanes and driven by a motor, causes the water in the central chamber to rotate as a mass at the rate of 240 R.P.M. With free communication between the central and outer chamber, *P*, how high will water stand in the open piezometer tube, *T* above the level in the reservoir? *Ans.* 11.8 ft.

9. A horizontal tube, 8 ft. long and 2 in. in diameter, is filled with water under a pressure of 10 lb. per square inch and closed at the ends. If

rotated in a horizontal plane about one end as an axis, at the rate of 60 R.P.M., what will the pressure at the outer end become?

Ans. 27.1 lb. per square inch.

10. A conical vessel with axis vertical and sides sloping at 30 degrees with the same is rotated about another axis distant 2 ft. from its own and parallel. How many revolutions per second must it make in order that water poured into it will be entirely discharged by the rotative effect?

Ans. 50.5 R.P.M.

11. A vertical cylinder 6 ft. high and 4 ft. in diameter, two-thirds full of water, is rotated uniformly about its axis until on the point of overflowing. (*a*) Compute the linear velocity at the circumference. (*b*) How fast will it have to rotate in order that 6 cu. ft. of water will be spilled?

12. A closed steel cylinder, 6 feet in diameter, 10 feet long, axis vertical, is just filled with water. How rapidly must it be rotated about its axis if the water pressure is to burst the sides of the cylinder by hoop tension? The metal is 0.25 inch thick and its ultimate strength is 50,000 lb. per sq. inch.

13. Prove that in the case of an overshot water-wheel revolving at uniform speed around a horizontal axis, the water surfaces in the various buckets will be cylindrical surfaces described from a common center at some point on the vertical diameter produced.

CHAPTER IV

FLUID MOTION—BERNOULLI'S THEOREM

33. WATER in motion presents problems more difficult and uncertain in their solution than those encountered in Hydrostatics, because of the presence of frictional resistances and other disturbances whose actions often defy mathematical expression. As a result, we are in many instances obliged to leave these forces out of our analysis and derive formulæ expressing laws of flow which are far from being satisfactory if viewed from the standpoint of the Analytical Mechanics, depending as they do upon carefully performed experiments to furnish constants or coefficients to make good the discrepancies which would otherwise exist between results indicated by theoretically deduced formulæ and those obtained from experiment. The nature of the difficulties which arise will be seen later.

34. Fluid Motion.—Straight Line and Sinuous Flow.—It is a fact well established by experiment that water in motion along any channel may flow in either of two widely differing ways, depending on the conditions present. At very low velocities the water particles appear to move in straight, parallel paths, the velocity of any given stream line being influenced by its nearness to the side of the channel which offers a resistance to its motion. At the wall, the velocity is zero and the maximum occurs at or near the center of the channel, depending on the shape of the channel's cross-section.

If the rate at which the water flows through the channel is made to gradually increase, no change in the type of motion will be noticed until a certain velocity known as the *Critical Velocity*, is reached. At velocities beyond this, the type of motion changes from straight line to *sinuous* or turbulent flow, in which the paths followed by the separate water particles become winding and intricately interlaced. Again, the maximum velocity of forward motion will be found near the center of the stream and there will

be a decrease in velocity as we approach the sides where it may or may not be zero, according to conditions present.

The two types of motion will be designated by the terms *straight-line* and *sinuous*, and it is very necessary to distinguish between the two as the laws governing them are totally different. The straight-line flow is relatively uncommon as in general it occurs at only very small velocities, much smaller than are usually met with in engineering practice. The slow movement of water through porous soils, underdrains and pipes of heating systems, illustrate possible cases of straight line motion.

Further discussion of these two types of motion and Critical Velocity will be found in Chapter VIII.

In the analysis of all problems presented in hydraulic flow it would be desirable and logical to consider and mathematically express the results of the different types of motion but, as already stated, sinuous motion defies exact mathematical analysis owing to our inability to definitely express the frictional forces present, their action, or their results. It has been found more practical, in a majority of cases, to assume a condition of straight-line flow and, having by theoretical discussion, found expressions for the laws of flow, modify our results by introducing into the expressions such numerical constants as *experimental* investigation may indicate.

35. Steady Flow.—If in any stream a plane section be taken at right angles to the direction of motion, the flow is said to be *Steady* when the *form of the cross-section*, the *velocity*, *pressure* and *fluid density* at that section remain constant. These quantities, save the last, may change in value from section to section but never at any one section. This is another way of stating that flow conditions are *permanent* and not pulsating. Such conditions are generally present in all hydraulic problems with which the engineer has to deal and, to problems of steady flow we shall limit our discussions.

Another helpful assumption, sometimes made, is that all particles move forward with the same velocity, so that at successive sections the same particles are found together. In reality this is never so, as the confining walls of the stream retard those particles near it and the sinuosity of the flow prevents any such condition.

36. Bernoulli's Theorem.—In 1738 Daniel Bernoulli demonstrated a general theorem in connection with fluid motion, which is probably the most important in the subject of Hydraulics. Upon it as a framework may be hung the whole fabric of Hydrokinetics, and by its use alone a great majority of the problems arising may be completely solved. In presenting it to the student it will be simpler if we first neglect all frictional resistances and then investigate their effect upon our result.

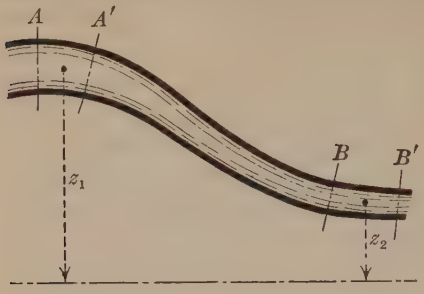


FIG. 39

(*a*) *Friction Neglected.*—Fig. 39 shows a portion of any stream or collection of stream lines. For convenience of thought and statement we may assume that the stream is confined within some pipe or conduit, so as to permit its being under a pressure which may change from section to section. We will further assume that the flow is *non-sinuous* and that all particles crossing a normal section move with the same velocity. The quantity per second, Q, passing any section must be constant, else there will be an accumulation of water. Let a_1 and v_1 represent respectively the cross-sectional area and velocity at A, and a_2 and v_2 the same quantities for a section at B. Then from the above,

$$Q = a_1 v_1 = a_2 v_2.$$

This is often referred to as the "*Equation of Continuity.*" Consider now the body of water contained between these sections and the forces acting upon it. (Fig. 40.) Those exerted by the pipe wall are normal to the sides of the stream (no tangential component since flow is frictionless) and do not concern us in the present discussion as will presently appear. At A and B the forces $a_1\ p_1$ and $a_2\ p_2$ represent the pressure of the rest of the water on the end sections. Really the pressure varies in its

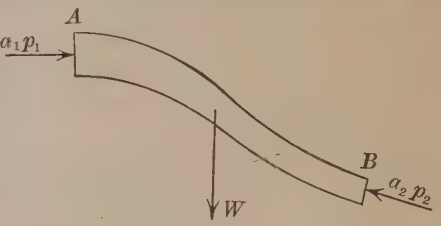


FIG. 40

intensity over each area, but if p_1 and p_2 represent the values of the average pressures, the forces are correctly stated. The only remaining force on the body is that exerted by gravity, W. By the principles of Mechanics, the work done in a given time by a system of forces acting on a body is equal to the change during that time in the kinetic energy of the body. Since a force must move in the direction of its line of application in order to do work, it will be seen that the normal pressures from the pipe do no work. Thus the working forces are three in number, a_1p_1, a_2p_2 and W.

Assume that in the short time, dt, A-B moves to A'-B' (Fig. 39). Then A moves to A', a distance equal to v_1dt, and B moves to B' a distance, v_2dt.

$$\text{Work done by } a_1p_1 = (a_1p_1)\ (v_1dt) = p_1\ Qdt.$$

$$\text{Work done by } a_2p_2 = -(a_2p_2)\ (v_2dt) = -p_2Qdt.$$

This last is negative, since it opposes motion. As for the work done by gravity during the move from A-B to A'-B', we may consider the change in the water's position to have been accomplished by moving A-A' to B-B', since the displacement of the center of gravity of the mass AB would be the same by either movement. The work done by gravity in making this change would be,

Work by Gravity $= wa_1\ (v_1dt)\ (z_1 - z_2) = wQdt\ (z_1 - z_2)$, z_1 and z_2 representing the heights, above *any* assumed datum plane, of the center of gravity of masses A-A' and B-B'.

We have now to find the change in kinetic energy of A-B. Conceive for a moment that all the particles of water between A and B be divided into a great number of groups of equal volume, the volume of each being that of A-A', which is the first group. Some of these are shown in Fig. 41, greatly exaggerated in size. With this assumption it is easily seen that as group a moves forward to displace b, in the time dt, its particles gradually assume the velocities that those in b had at the beginning of the time dt, and thus the group experiences a change in kinetic energy. Simultaneously with this movement, each group from a to m advances into the position previously held by the immediately

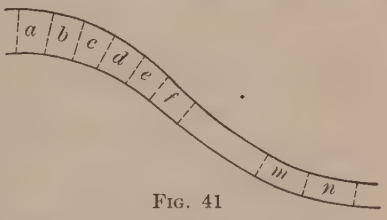


FIG. 41

preceding group, and m moves to B-B' at n. The change in energy experienced by any group, as c, would be the difference between the original kinetic energies of c and d, inasmuch as c acquires d's velocity and (having the same mass) its kinetic energy. The total change in kinetic energy for the entire body A-B would be obtained therefore by adding the differences between contiguous groups. There results:—

Total change in $K.\,E.$ =

$$(k_b - k_a) + (k_c - k_b) + (k_d - k_c) + \text{etc.} \ldots (k_n - k_m),$$

or, $$\text{Change in } K.\,E. = k_n - k_a.$$

The change is then equal to the difference between the kinetic energy of A-A' and B-B', or,

$$\text{Change in } K.\,E. = \frac{w.Q.dt}{2g}(v_2^2 - v_1^2).$$

Finally, equating work done to change in energy we have

$$p_1Qdt - p_2Qdt + wQdt\,(z_1 - z_2) = \frac{wQdt}{2g}(v_2^2 - v_1^2), \quad . \quad (29)$$

or,
$$\frac{v_1^2}{2g} + \frac{p_1}{w} + z_1 = \frac{v_2^2}{2g} + \frac{p_2}{w} + z_2. \quad . \quad . \quad . \quad (30)$$

This expression constitutes Bernoulli's Theorem. It states that under steady flow, with friction eliminated,

$$\frac{v^2}{2g} + \frac{p}{w} + z = \text{a constant quantity.} \quad . \quad . \quad . \quad (31)$$

for all sections.

Each of these three terms is a *linear distance.* The value $\frac{p}{w}$ we are already familiar with, it being the head that gives rise to the pressure p, and known as *Pressure-Head.* The term z is simply the height of the particle above *any* assumed datum and will be known as the *Potential Head.* Lastly, $v^2 \div 2g$ is *Velocity Head.* The significance of the name will appear shortly. The sum of these three heads for any particle we will call the *Total Head.*

Our last equation may now be thus expressed in words: *In steady flow, without friction, the sum of velocity head, pressure head, and potential head is a constant quantity for any particle throughout its course.*

(*b*) *Friction Considered.*—Friction between the stream and the channel walls and also between adjacent water particles, produces resistances or forces which hinder motion. The work of such forces is therefore negative, and equation (29) must be changed by adding to the left-hand member a negative quantity to represent this work. The quantity must contain a force and a distance, and we may employ any force we please provided a proper distance be used. Selecting the force $w.Qdt$, which already appears in the equation, and representing the distance by h', we have

$$p_1Qdt - p_2Qdt + wQdt\,(z_1 - z_2) - wQdt\,h' = \frac{wQdt}{2g}(v_2^2 - v_1^2),$$

or
$$\frac{v_1^2}{2g} + \frac{p_1}{w} + z_1 - h' = \frac{v_2^2}{2g} + \frac{p_2}{w} + z_2, \quad . \quad . \quad . \quad (32)$$

or
$$h' = \left(\frac{v_1^2}{2g} + \frac{p_1}{w} + z_1\right) - \left(\frac{v_2^2}{2g} + \frac{p_2}{w} + z_2\right). \quad . \quad . \quad (33)$$

Here in (33) h' is seen to be the difference between the totals of the heads at the two sections. We may then call it the Lost Head due to friction (sometimes spoken of as "Friction Head"). Again, equation (33) may be thus expressed in words:—*In steady flow, with friction, the total head at any section is equal to that at any subsequent section plus the lost head occurring between the two sections.*

In the foregoing demonstrations it was assumed that the velocity of all particles crossing any one section was the same or, in other words, the particles adjacent to each other continued to hold their relative positions as they moved forward to another section. It has been pointed out that this is almost never the case since friction on the sides of the pipe or channel causes those particles nearer it to be retarded. The intensity of the pressure at any section was assumed to be p, but we shall later prove that it varies practically uniformly over the section; also the values of z for the several particles in the section differ according to their position. Strictly speaking, our theorem applies only to the theoretical stream line followed by a single particle, but if the *average* values of velocity, pressure and potential head at any section be used when making numerical substitutions in the theorem, the validity of the latter is not impaired.

37. Application of Bernoulli's Theorem.—In a majority of the problems arising in practice, it is the velocity or pressure at a certain point that is desired. In experimental work more often it is the value of the lost head that is unknown and sought. Since there are seven heads appearing in the equation, and only one, in general, can be unknown, it is necessary in choosing the points or sections between which we write the theorem, *to find points at which all the heads, save the one desired, are known.* In most cases, as we shall see later, this is possible. Often where there appear to be two unknowns, a second equation may be written giving a relation between them. Thus if one velocity head be unknown, it is often possible to express it in terms of the velocity of another point from the relation

$$a_1v_1 = a_2v_2 = a_3v_3, \text{ etc.}$$

Later we shall see that it is often possible to express the lost head as a function of one or both of the velocity heads.

It should be especially noted here that the solution of the equation for any unknown term will *not* result in a *real* value unless the term representing the lost head be incorporated in it. Often it becomes necessary to temporarily solve the equation without the use of the lost head and the "ideal" or "unreal" value so obtained must be then corrected by methods which will be given later. The student is advised to early acquire the power of applying this important theorem with confidence and understanding, as it will prove to be the key to many an otherwise difficult or lengthy problem.

Example.—A pipe line gradually enlarges from 24 in. in diameter at A to 36 in. at B. The velocity at A is 5 ft. per second, and the average pressure, above that of the atmosphere, 50 lb. per square inch. Assuming that 2 ft. of head are lost between A and B, find the pressure at the latter point if it be situated 15 ft. lower in elevation than A. (The student should draw the figure.)

From Bernoulli's Theorem,

$$\frac{v_1^2}{2g} + \frac{p_1}{w} + z_1 = \frac{v_2^2}{2g} + \frac{p_2}{w} + z_2 + \text{Lost head.}$$

From the conditions stated,

$$v_1 = 5 \text{ ft. per second.}$$
$$p_1 = 50 \text{ lb. per square inch.}$$
$$z_1 = 15 \text{ ft.}$$
$$z_2 = 0 \text{ ft.}$$
$$\text{Lost head} = 2 \text{ ft.}$$

(It will be noted that the datum from which the z's were reckoned was assumed passing through the point B.) As for the value of v_2, we have

$$a_1 v_1 = a_2 v_2,$$

giving

$$v_2 = \frac{a_1 v_1}{a_2} = \left(\frac{24}{36}\right)^2 \times 5 = 2.22 \text{ ft. per second.}$$

Substitution of the above values gives

$$\frac{25}{65.4} + \frac{50 \times 144}{62.5} + 15 = \frac{4.93}{64.4} + \frac{p}{w} + 0 + 2.$$

$$\frac{p}{w} = 128.4 \text{ ft.}$$

$p = 8025$ lb. per square foot, or 55.7 lb. per square inch. *Ans.*

38. Pressure and Its Measurement.—In Article 9 it was pointed out that pressure intensity may be measured with the atmospheric pressure as a starting point (Relative Pressure) or so as to include the latter (Absolute Pressure). Obviously either scale of measurement may be used in writing the value of the pressure-head in Bernoulli's Theorem since the change from one scale to the other involves simply the addition (or subtraction) of 34 feet to or from each pressure-head and hence to or from each side of the equation. *The same scale should always be employed in any one writing of the equation.*

The value of the pressure intensity may be anything from absolute zero upward, but it can never be less than zero as a negative value would imply tension, which water cannot stand (save in so slight a degree as to be negligible).

If in the solution of the general equation a negative value of p be obtained, it may at once be known that either an error has been made or that flow does not take place under the conditions assumed. If a tendency to cause negative pressure does exist,

the stream must break apart at such a point, and the tube or channel will be filled with water vapor.

All of the piezometers and gauges described in Arts. 12 to 15 may be used for determining pressures or differences in pressure existing in pipes containing water in motion. The water in the piezometer columns is *at rest* and hence their heights must measure exactly the pressure at their bases. That the pressure at a point *in* the moving stream, directly beneath the column, is correctly measured by the height of the column above that point, may be seen if we consider the motion of the elementary prism of water directly under the still column, and the forces acting on it. Since it is moving in a direction at right angles to the piezometer column, it has no acceleration toward or away from the column and the algebraic sum of the vertical forces must equal zero. Reference to Fig. 42, in which are shown the forces acting on the prism, enables us to write

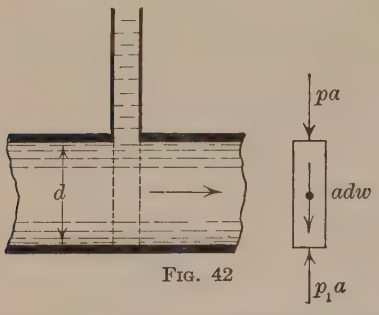


FIG. 42

$$p_1 a = pa + adw$$

(a being the horizontal, sectional area of the prism.)

or
$$\frac{p_1}{w} = \frac{p}{w} + d,$$

from which it appears that the pressure varies uniformly from the top to the bottom of the prism as it would if the latter were at rest It follows that the average pressure should be found at the center of gravity of the section, which is the same point as used for computing the average potential head at the section.

In attaching a piezometer of any kind to a pipe, it is important that certain precautions be taken regarding the manner of inserting the connecting tube into the pipe wall.

Figure 43 shows three tubes, of which (a) is normal to, and has its inner edge flush with, the surface of the pipe wall. It measures the pressure at its base correctly. Both (b) and (c) are inclined to the inner surface, although their ends are flush with it. The

water in (*b*) will stand lower than in (*a*), while in (*c*) it will stand higher. Both give incorrect readings on account of their columns being affected by the velocity of the moving water. Had their lower ends not been flush with the inner surface, the discrepancies would have been further magnified. In the case of the tube (*a*) it has been supposed that cohesion might cause some of the water to be drawn out of the tube and the level to be lowered. Careful experiments have shown that such is not the case if care be taken to make the tube normal to, and flush with, the surface.

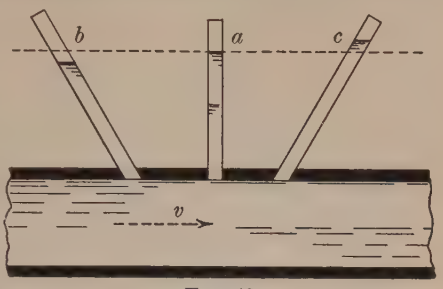


FIG. 43

39. Bernoulli's Theorem as an Equation in Energy.—A body moving through space, unhindered by friction, has a *constant* stock of energy which exists partly as *potential* and partly as *kinetic* energy. Thus a body may start from a state of rest with only potential energy, and, having fallen through a height h, acquire a kinetic energy which is the exact equal of the potential energy lost in that distance. If at this instant it be a distance z above some horizontal plane of reference, its total stock of energy must be

$$E = Wz + \frac{Mv^2}{2} = W\left(z + \frac{v^2}{2g}\right),$$

or, *per pound of substance,*

$$E = z + \frac{v^2}{2g}.$$

With a mass of moving water (friction eliminated) it is also true that its stock of potential and kinetic energy is constant, if we consider the *entire mass* concerned. If, however, we consider a small portion of it only, we find that such is not the case. Thus in a pipe of constant diameter it will be seen that, if the pipe is not horizontal, a small mass may be constantly losing potential energy as it progresses and yet receive no addition to its kinetic energy (velocity the same at all points). If its total stock of energy is to remain constant (friction eliminated), we must recognize the

existence of another form of energy which we may, for want of a better name, call *pressure energy*. This we will define as the ability of the water to do work by virtue of the pressure derived from its contact with adjacent masses. That this ability does exist may be shown by discussing the action of a very simple water motor. We will imagine a closed horizontal cylinder of small cross-section and length l, fitted with a piston and communicating by means of properly arranged valves with a large reservoir of water. The piston being at one end of its stroke, the opening of a valve admits water under a pressure p, and the piston moves to the other end of the cylinder. The valve being now closed and an exhaust port opened, the contained water is driven from the cylinder without resistance by a return movement of the piston now receiving water on its other face from the opposite valve. The work of a single stroke is $A \cdot p \cdot l$, and the weight of the water used is $A \cdot l \cdot w$. We may then say that, *per pound of water used*,

$$\text{Work done} = \frac{Apl}{Alw} = \frac{p}{w} \text{ ft. lb.}$$

While in the motor, the water gave up none of its potential energy (motion horizontal), neither did it give up kinetic energy, since its velocity at the beginning and at the end of the stroke was zero. The work done by the water, therefore, was at the expense of its pressure, which fell from p to zero (relative pressures) as it passed from the cylinder. It should be noted that the water used depended for its energy on its connection with a mass of water having a higher potential; and had it not been for the presence of the latter, no work could have been done. Pressure energy may be said to be the existence of potential energy in another form, and in the case of the small mass of water contained in the pipe mentioned at the beginning of this article, a loss in potential is balanced by a gain in pressure energy. We may write, for any position of the mass,

$$\frac{v^2}{2g} + \frac{p}{w} + z = \text{a constant}, \quad . \quad . \quad . \quad . \quad (34)$$

the three terms representing, respectively, the kinetic, pressure, and potential energy of the mass *per pound of weight*. We recognize in this equation Bernoulli's Theorem, and see that the latter is really an expression of the principle of Conservation of Energy.

If the movement of the water is accompanied by friction, the sum of the three energies is not constant and between any two points in the stream we should have

$$\frac{v_1^2}{2g} + \frac{p_1}{w} + z_1 = \frac{v_2^2}{2g} + \frac{p_2}{w} + z_2 + \text{Lost energy per lb.}$$

We have, therefore, two conceptions of the meaning of the terms in Bernoulli's Equation. As first viewed, they represented distances or *Heads;* now we see they also represent energy *per pound* of liquid. The two conceptions are synonymous since

$$\text{Energy per pound} = \frac{\text{Foot-pounds}}{\text{pound}} = \frac{\text{Force} \times \text{Distance}}{\text{Force}}$$
$$= \text{Distance or Head}$$

This statement must be mastered by the student before going further in his study.

The energy, per pound of water, passing a section being given by (34), the total *energy per second* passing the section would be obtained by multiplying by W, the weight of water passing the section per second. Thus,

$$\text{Total Energy per second} = W\left(\frac{v^2}{2g} + \frac{p}{w} + z\right) \text{ ft. lb. per second,}$$

and comparing the energy at any two sections, we have

$$W\left(\frac{v_1^2}{2g} + \frac{p_1}{w} + z_1\right) = W\left(\frac{v_2^2}{2g} + \frac{p_2}{w} + z_2\right) + \text{Lost energy per second.}$$

This latter equation is of great value in determining the amount of energy absorbed by a water-turbine or derived from a pump. In either case it is only necessary to compute the energy per second in the stream at points just before and after passing the machine, and the difference in the two quantities must have gone into the turbine or been derived from the pump.

In the case of a stream formed free in the air, the energy at any point must be wholly kinetic (datum through the point) and equal to $W\frac{v^2}{2g}$ ft. lb. per second, which is the same as $\frac{Mv^2}{2}$, the usual expression for kinetic energy in a moving mass. It should be noted that hydraulic energy is necessarily computed in foot-pounds *per second*, whereas the energy of a moving *rigid* body is computed in foot-pounds.

Example.—A 12-inch pipe-line contains water moving with a mean velocity of 10 feet per second. What energy per second passes a section where the average pressure is 20 lb. per sq. inch?

$$\frac{v^2}{2g} = \frac{100}{64.4} = 1.56 \text{ ft. lb. per lb.}$$

$$\frac{p}{w} = 20 \times 2.3 = 46 \text{ ft. lb. per lb.}$$

Total Energy per lb. = 47.6 ft. lb. per lb.

The pounds per second passing the section are

$$W = .785 \times 10 \times 62.5 = 491 \text{ lb.}$$

$\therefore$ Energy per second $= 47.6 \times 491 = 23{,}370$ ft. lb.

Expressed in Horse Power, this is equivalent to

$$\frac{23{,}370}{550} = 42.5 \text{ H. P.}$$

It will be noticed that potential energy was not considered for the simple reason that we were computing the energy *in* the pipe and the reference datum was, therefore, through the pipe's axis. Had it been required to ascertain the Energy with reference to a datum anywhere below (or above) the pipe, the potential energy per pound would, of course, have been algebraically added to the above sum.

Example.—Water enters a motor through a 2-foot pipe under a gauge pressure of 20 lb. per square inch. It leaves by a 3-foot pipe under a gauge pressure of 5 lb. per square inch. A vertical distance of 6 feet separates the centers of the two pipes at the sections where the pressures are measured. If 15.7 cu. ft. of water pass the motor each second, compute the rate at which energy is given up by the water to the motor.

At entrance,

$$v_1 = \frac{15.7}{3.14} = 5 \text{ ft. p. s.}$$

At exit,

$$v_2 = \frac{15.7}{7.07} = 2.2 \text{ ft. p. s.}$$

Applying Bernoulli's Theorem to the stream at these points,

$$\frac{(5)^2}{64.4} + \frac{20 \times 144}{62.5} + 6 - \text{Energy given up to the motor by each pound} = \frac{(2.2)^2}{64.4} + \frac{5 \times 144}{62.5} + 0$$

From which

$$E = 52.4 - 11.6 = 40.8 \text{ ft. lb. per lb.}$$

Energy furnished Motor each second is

$$E = (15.7 \times 62.5)\ 40.8 = 40{,}000 \text{ ft. lb. per second}$$

This is equivalent to $\frac{40000}{550}$ or 72.7 Horse Power

PROBLEMS

1. An 8-inch pipe contains a short section in which the diameter is gradually reduced to 3 in. and then gradually enlarged to full size. If the pressure of water passing through it is 75 lb. per square inch at a point just before the reduction commences, what will it be at the 3-inch section when the rate of flow is 1.20 cu. ft. per second? Assume no loss in head between the two sections considered.

Ans. 71.0 lb. per square inch.

2. A 3-inch pipe discharges into the air at a point 6 ft. above the level ground, the water leaving the pipe with a velocity of 28 ft. per second. Assuming air friction to be negligible, compute the velocity of the water as it strikes the ground if, (*a*) the pipe be horizontal; (*b*) the pipe be inclined upward 45 degrees from the horizontal.

Ans. 34.2 ft. per second.

3. A 12-inch pipe discharges water at the rate of 5.5 cu. ft. per second. At a section, *A*, on the pipe, the pressure is 40 lb. per sq. inch while at section *B*, at a point where the pipe is 8 ft. lower than at *A*, the pressure is 42.5 lb. per sq. inch. Compute the head lost between *A* and *B*.

4. Water flows radially outward in all directions from between two horizontal circular plates which are 4 feet in diameter and placed parallel 1 inch apart. A supply of 1 cubic foot per second being maintained by a pipe entering one of the plates at its center, what pressure will exist between the plates at a point 6 inches from the center if no loss by friction be considered?

(NOTE.—The student may verify his answer by cutting two small disks from cardboard, piercing one at its center by a small pipe or piece of stiff straw, and blowing air into the pipe instead of water.)

Ans. $p = 0.1$ lb. per square inch below atmosphere.

5. A horizontal pipe 12 in. in diameter carries water with a mean velocity of 10 ft. per second. At a section, *A*, the pressure is 55 lb. per

square inch and at a section, B, it is 40 lb. per square inch. Compute the quantity of energy passing each of these sections in 1 second, estimating the potential energy with reference to a datum plane through the pipe's axis. What is the amount of head lost between the two sections?

Ans. (*a*) 62,900 ft. lb.
(*b*) 45,900 ft. lb.
(*c*) 34.5 ft.

6. A 2-inch stream of water issues from a nozzle with a velocity of 75 ft. per second. What quantity of energy passes the nozzle per second if the datum plane for computing potential energy be taken through the axis of the issuing stream?

Ans. 8930 ft. lb.

7. Water from a reservoir is pumped over a hill through a pipe 3 ft. in diameter, and a pressure of 30 lb. per square inch is maintained at the summit, where the pipe is 300 ft. above the reservoir. The quantity pumped is 49.5 cu. ft. per second and by reason of friction in the pump and pipe there is 10 ft. of head lost between reservoir and summit. What amount of energy must be furnished the water each second by the pump?

Ans. 2134 H. P.

8. How much energy, in horse-power units, is being transmitted through a 3-inch pipe in which the velocity of flow is 15 ft. per second and the gauge pressure 40 lb. per square inch? If a section farther on in the pipe were considered, would there be a less amount of energy passing each second? What form of energy would remain constant?

Ans. 8 H. P.

9. In order to maintain a discharge of 1.20 cu. ft. per second through a 6-inch pipe which discharges into the air, it is found necessary to keep a pressure of 36 lb. per square inch in the pipe at the inlet end, which is 8 ft. above the discharge point. Compute the loss in head while passing through the pipe. How much energy per second does it represent?

Ans. 90.8 ft.
6810 ft. lb. per second.

10. A 2-inch vertical pipe, supplied from a reservoir, discharges 0.22 cu. ft. per second into the air, and it is noted that its cross-section is filled by the water at exit. What will be the pressure in the pipe at a point 12 ft. above its lower end? Where will the pressure be absolute zero? What will it be at a point 40 ft. above the end? What fact is implied by the numerical value of the last result? Neglect friction in all cases.

Ans. (*a*) $p = 5.2$ lb. per square inch below atmospheric pressure.
(*b*) 34 ft. above end.

11. Water enters a motor through a 4-inch pipe under a gauge pressure of 150 lb. per square inch. It leaves by an 8-inch pipe at an elevation 3 ft. below the point of entrance. If the pressure in the pipe at exit be 10 lb. per square inch and the discharge 2 cu. ft. per second, find the energy given up by the water each second as it passes the motor.

Ans. 41,600 ft. lb.

12. During the test of a centrifugal pump, a gauge just outside the casing and on the 8-inch suction pipe registered a pressure 4 lb. per square inch less than atmospheric. On the 6-inch discharge pipe another gauge indicated a pressure of 30 lb. per square inch above atmospheric.

If a vertical distance of 3 ft. intervened between the pipe centers at the sections where the gauges were attached, what horse power was expended by the pump in useful work when pumping 2 cu. ft. per second?

Ans. 18.7 H. P.

CHAPTER V

DISCHARGE FROM ORIFICES

40. General Remarks.—As commonly used in hydraulics, the term, orifice, applies to any opening having a closed perimeter, made in a wall or partition, thereby permitting water held under pressure within the wall to escape. Orifices enter into the design of many hydraulic structures and are often used for the measurement of flowing water. They differ not only in geometrical shape but also in the manner in which their edges or perimeters are formed.

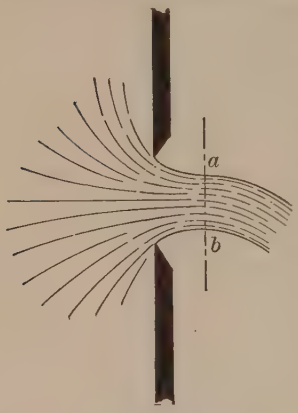


FIG. 44

The circular and rectangular orifice are most commonly used in engineering construction and upon these, especially the former, much experimental work has been done. This has been limited in a majority of cases to orifices having a *sharp edge* or an edge so thin that the stream in passing it touches only a line (see Fig. 44). The rate of discharge from an orifice depends to a considerable extent upon the nature of its edge and in order to compare the performances of orifices having different diameters it is necessary that their edges be similarly formed. Inasmuch as the variety of edges is practically unlimited, it has been generally agreed to make the sharp-edged orifice the standard for comparison. It may be fashioned as in Fig. 44 or it may have the form shown in Fig. 50*b* where a plate or wall of small measurable thickness has been pierced by a hole and a sharp, well defined edge produced at the inner surface of the plate. Experiment shows that there is no difference in the hydraulic properties of these two orifices.

41. Flow from a Sharp-edged Orifice.—Water flowing from an orifice having a sharp edge presents the following characteristics. As the stream leaves the orifice it gradually contracts to form a jet

whose cross-sectional area is somewhat less than that of the orifice (Fig. 44). This is doubtless due to the convergency of the paths followed by separate particles as they approach the orifice and to the motion along the wall which particles close to the wall had as they approached the orifice. The contraction is not complete until the section *a-b* is reached. For a circular orifice, having its edge in the same plane as the inner face of the wall, *a-b* is distant from that plane approximately one-half the diameter of the orifice. At this point in the jet and at any subsequent section, the pressure throughout the section is that due to the surrounding atmosphere, and in general this is true for any jet formed free in air. If any small mass or portion of the jet beyond *a-b* be considered, it will be seen that the particles which form it are falling freely through the air without horizontal acceleration (resistance due to atmosphere neglected). The only force acting on the mass is that of gravity, and by Art. 29 it follows that the pressure in the mass is atmospheric.

In the short portion of the jet between the edge of the orifice and the plane *a-b*, the pressure will be greater than atmospheric since the particles are moving in curved paths and must be acted upon by centripetal pressures of greater intensity than that of the atmosphere. The writing of Bernoulli's Equation between two points, one in the plane of the orifice, the other in the plane, *a-b*, will establish the same fact. Since potential heads are alike and the velocity head at the first named point is less than at the second, it follows that the pressure-head at the orifice is greater than at the contracted section.

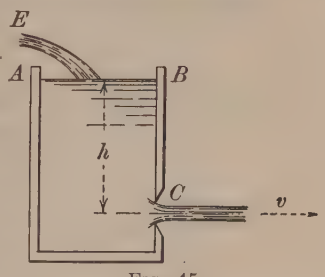


FIG. 45.

Let Fig. 45 represent a sharp-edged orifice in the side of a large tank or reservoir, having a depth or head of water, *h*, on its center. Since this head is maintained constant by the inflow at *E*, the case is one of steady flow. The area of the reservoir surface will be assumed to be very large compared with the area of the orifice, the result being that the particles in the surface will have no appreciable downward *Velocity of approach*. This being so, their velocity *head* may be taken as zero. Eliminating for the moment all friction, we will apply Bernoulli's Theorem to a par-

ticle in the surface A-B, and in its later position C, at the contracted section of the jet.

The hydrostatic pressure-head when at the first position is zero in value. Assuming the datum plane through the center of the orifice, the potential head will be h. At C the velocity head is $\frac{v^2}{2g}$, the pressure-head is again zero and the potential head also zero.

There results

$$0 + 0 + h = \frac{v^2}{2g} + 0 + 0$$

or

$$v = \sqrt{2gh}. \qquad (35)$$

which we may call the *theoretic* or *ideal* value of the velocity, friction having been neglected. Several important deductions follow. Under the assumed conditions the water would attain a velocity equal to that which it would have if it had fallen freely through a height h.

Using the first obtained relation,

$$h = \frac{v^2}{2g}$$

it will now be seen why in Art. 36 we gave the name, *Velocity Head* to the term $\frac{v^2}{2g}$, it being the head which would produce the velocity v. Whenever it subsequently appears it will be so designated.

It might be here noted that the orifice shown in Fig. 45 is in a vertical plane and, therefore, is exposed to a head which is different at different parts of the orifice. The issuing jet will, therefore, be made up of particles or stream lines which will have velocities varying as the *square roots* of the heads which produced them and the value of v as obtained from $v = \sqrt{2gh}$ will not be a true mean for all the particles as they pass the contracted section. It would be if the velocities varied *directly* as the heads, since the head on the center is the mean head. If v varies as $\sqrt{h}$, it will be seen that the curve in Fig. 47 showing this variation, is a parabola with vertex in the water surface and axis vertical. It is obvious from the

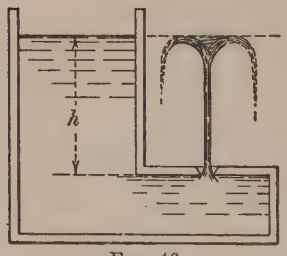


FIG. 46

figure that the variation in velocity throughout the cross-section of an orifice will be greater as h decreases, and for very low heads the true mean velocity cannot be computed from $v = \sqrt{2gh}$. However, if the head be large in proportion to the vertical dimension of the orifice, the true mean will differ but a small fraction of one per cent. from $\sqrt{2gh}$ and no sensible error will result.

With an orifice in a *horizontal* plane, all parts of it will be under the same head and the velocity of all the particles would be alike on arriving at the contracted section.

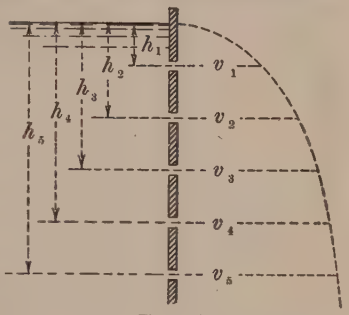


FIG. 47

Since the ideal velocity due to a head is the same as though the water particles had fallen freely through the same height, we should expect that, if the orifice were horizontal and the jet directed upward (Fig. 46), the stream would rise to a height equal to the head which produced it. Of course air resistance would prevent this from actually happening, as would friction at the orifice and between the water particles. Experiment has shown that under low heads (less than six or eight feet), the discrepancy is very slight but becomes more marked as the head increases.

42. Coefficient of Velocity.—Experiment shows that the actual velocity of a jet from a sharp-edged orifice is a little less than that found from $v = \sqrt{2gh}$, mostly because of friction at the edge of the orifice, but also because water is not a perfect fluid. The loss in velocity due to friction is about two per cent in this case, giving as the value for actual velocity,

$$\text{Actual Velocity} = 0.98\sqrt{2gh} = c_v\sqrt{2gh}, \quad . \quad . \quad (36)$$

where 0.98 is called the *Coefficient of Velocity*. For standard orifices it varies little in a range of several hundred feet of head, so we may use the above value in all numerical work.

43. Coefficient of Contraction.—The ratio of the area of the contracted section of the jet to that of the orifice is designated as the *Coefficient of Contraction*. It varies slightly with the size of

the orifice and with the head, having a mean value of about 0.62 for standard orifices.

$$\therefore c_c = \frac{a'}{a} = 0.62 \text{ approx. for standard orifices (Fig. 48).}$$

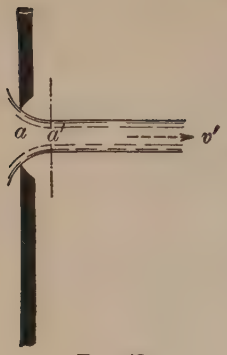


FIG. 48

Its chief use is in determining the area of the jet at the contracted section.

44. Coefficient of Discharge.—The quantity or volume Q, flowing from the orifice per second may be stated as the product of a', the actual area of the contracted section, and the actual velocity past that section. Therefore we may write

$$Q = a'v' = ac_c v' = ac_c c_v \sqrt{2gh},$$

or
$$Q = c_d a \sqrt{2gh}, \quad . \quad . \quad . \quad . \quad . \quad (37)$$

where $a\sqrt{2gh}$ represents the *ideal* discharge which would have taken place had no friction or contraction been present. As for c_d, it is the coefficient by which we multiply the *ideal* discharge in order to obtain the actual and, we shall call it the *Coefficient of Discharge.* Numerically it is always equal to $c_v \times c_c$ and for a standard orifice its mean value will therefore be

$$c_d = c_v \times c_c = 0.62 \times 0.98 = 0.61$$

For strictly accurate determination of discharge its exact value should be determined for each particular orifice and head.

45. Determination of the Value of the Coefficients.—(*a*) *Coefficient of Contraction.* The only direct way of measuring the amount of contraction is to caliper the contracted section. This may be done by inclosing the jet with a ring containing finely pointed screws, set radially in its circumference, which can be adjusted so as to just touch the jet as it passes the ring. The latter may then be removed and, if the screws be set on diameters, these may be measured and a mean diameter obtained. The method is not wholly satisfactory, as it is difficult to keep the jet steady. Another method consists in first determining the coefficients of velocity and discharge, and then using the relation

$$c_c = c_d \div c_v.$$

(*b*) *Coefficient of Velocity.* This is generally obtained by making a series of measurements on the path of the jet. If a particle issues from the orifice with a velocity v, and in t seconds is found at the point m (Fig. 49), we may write

$$y = vt,$$

$$\text{and } x = \frac{gt^2}{2} \text{ (freely falling body).}$$

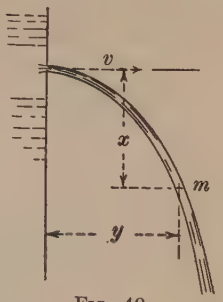


FIG. 49

Between these two equations we may eliminate t, obtaining

$$y^2 = \frac{2v^2}{g}x,$$

which shows the path to be a parabola with vertex at the orifice. Evidently if we measure the x and y of any point in the jet, we may use the above equation to determine the actual velocity of flow from the orifice. The coefficient is then obtained from

$$c_v = \frac{\text{Actual velocity}}{\text{Ideal velocity}}.$$

The errors present in this mode of determination arise not only from inaccurate measurement of x and y, but from the fact that the atmosphere retards the particles after leaving the orifice. The values of x and y are obtained by placing a ring in a supporting frame, and adjusting the whole so as to cause the jet to pass through the ring. This done, the flow is stopped and the coördinates of the ring measured. This may be done at several different points and a mean value for v obtained. With the exercise of care, it is probable that a very good determination of the coefficient may be made.

(*c*) *Coefficient of Discharge.* This may be more easily found than the others, since it is only necessary to allow the orifice to discharge for a known length of time and measure (by volume or weight) the amount of water passed. The actual and ideal quantities may then be compared for the coefficient.

The values of c_v and c_c are seldom required by the engineer, and it is only c_d (or simply c, as we shall hereafter call it) that is of interest to him. To determine this, thousands of experiments have been made in order that its value may be known for a wide range of heads and sizes of orifice.

46. Values of the Coefficients.—It has been stated that the value of c_v is nearly constant at 0.98 for all sharp-edged or standard orifices, irrespective of their shape or the head upon them. For other orifices it is probable that small error will result from assuming c_v to have the same value.

No tabulated set of values for c_c is available and for standard orifices it is generally customary to use values obtained from the relation

$$c_c = c \div c_v,$$

using values for c as given below.

The most widely quoted values for the Coefficient of Discharge, c, are those which appeared in 1886 in Hamilton Smith's "Hydraulics." They were based on experiments made by himself and have been in quite general use for many years. They are given in the following table and it may be noted that they decrease in value as the head and diameter of orifice increase, reaching a common value under a head of 100 feet. Just why they should do this latter thing is not quite apparent and in the light of experiments more recently performed some doubt has arisen regarding their accuracy.

COEFFICIENTS OF DISCHARGE (c) FOR CIRCULAR ORIFICES

From Hamilton Smith's "Hydraulics"

HEAD h IN FEET	DIAMETER OF ORIFICE IN FEET						
	0.02	0.04	0.07	0.1	0.2	0.6	1.0
0.4		0.637	0.624	0.618			
0.6	0.655	.630	.618	.613	0.601	0.593	
0.8	.648	.626	.615	.610	.601	.594	0.590
1.0	.644	.623	.612	.608	.600	.595	.591
1.5	.637	.618	.608	.605	.600	.596	.593
2.0	.632	.614	.607	.604	.599	.597	.595
2.5	.629	.612	.605	.603	.599	.598	.596
3.0	.627	.611	.604	.603	.599	.598	.597
4.0	.623	.609	.603	.602	.599	.597	.596
6.0	.618	.607	.602	.600	.598	597	.596
8.0	.614	.605	.601	.600	.598	.596	.596
10.0	.611	.603	.599	.598	.597	.596	.595
20.0	.601	.599	.597	.596	.596	.596	.594
50.0	.596	.595	.594	.594	.594	.594	.593
100.0	.593	.592	.592	.592	.592	.592	.592

Messrs. Judd and King of the Ohio State University, experimented on small circular orifices, ranging from ¾ inch to 2½ inches in diameter, and found very little change in the coefficient for heads ranging from 4 to 93 feet. Their results are summarized in the following table, and a complete account of their work appears in "The Engineering News," Sept. 27, 1906.

COEFFICIENTS OF DISCHARGE

From Judd and King

DIAMETER IN INCHES	VALUE OF c
¾	0.6111
1	0.6097
1½	0.6085
2	0.6083

These values are seen to decrease as the diameter increases but they were practically independent of the head when the latter exceeded 4 feet.

In 1908 Mr. H. J. Bilton of Australia published in "The Engineer" an account of some experiments upon small circular orifices, from which, and those of Judd and King, it would appear that for small orifices, up to 2½ inches in diameter, each size of orifice has a critical head above which there occurs no variation in c. The value of this critical head as determined by Bilton, together with values of c, appear below.

COEFFICIENTS OF DISCHARGE

By Bilton

HEAD IN INCHES	DIAMETER OF ORIFICE IN INCHES						
	0.25	0.50	0.75	1	1.5	2	2.5
3	0.680	0.657	0.646	0.640			
6	0.669	0.643	0.632	0.626	0.618	0.612	0.610
9	0.660	0.637	0.623	0.619	0.612	0.606	0.604
12	0.653	0.630	0.618	0.612	0.606	0.601	0.600
17	0.645	0.625	0.614	0.608	0.603	0.599	0.598
18	0.643	0.623	0.613				
22	0.638	0.621					
45	0.628						

The last figure in each column represents the value of c which appeared constant for heads above the critical head given in the left-hand column opposite the coefficient.

The values agree very well with those of Judd and King and may be regarded as accurate to every practical degree.

Mr. A. A. Barnes, of Manchester, England, has proposed for circular orifices,

$$Q = 3.73\ D^{1.985}\sqrt{H}. \quad . \quad . \quad . \quad . \quad (38)$$

This formula, rational in form but empirical in its two numerical quantities, is based on a study of the above mentioned experiments and others. Its results, when compared with known experimental values, indicate it has a high degree of accuracy and, while the comparison has been made only up to diameters of 12 inches, there seems to be no reason why it should not apply to any diameter, or any head above 1.50 feet. The main deterrent in its use is the odd exponent of D which necessitates logarithms. This small disadvantage disappears when computing discharges from a given orifice, as the equation is simplified to

$$Q = K\sqrt{H}.$$

Example.—Required the velocity of flow and volume discharged per second from a standard, circular orifice 2.50 inches in diameter under a head of 9 feet.

By equation (36),

$$v = 0.98\sqrt{2 \times 32.2 \times 9} = 23.6 \text{ ft. per second}$$

By equation (37), using Smith's coefficient,

$$Q = 0.5975 \times 0.034\sqrt{2 \times 32.2 \times 9} = 0.488 \text{ cu. ft. per sec.}$$

Using Judd and King's coefficient,

$$Q = 0.608 \times 0.034\sqrt{2 \times 32.2 \times 9} = 0.497 \text{ cu. ft. per sec.}$$

Using Bilton's coefficient of 0.598, the result is the same as by Smith's coefficient, while the use of Barnes' empirical formula gives the same result as obtained by Judd and King's coefficient.

47. Coefficients for Other than Standard Orifices.—Orifices are commonly employed in the design of many hydraulic constructions, such as dams, sluices, gates, headworks, etc., and in parts of many hydraulic machines. Their form is such that they can rarely

be considered as *standard* orifices, and the probable value of their coefficients becomes of importance. A few cases of actual determination of the value of c exist, and reference to these may be found in various articles of hydraulic literature. Generally it remains for the engineer to estimate its probable value in each case and the very best judgment and experience is then required. Any departure from a sharp edge will affect the amount of the contraction and consequently the discharge. Figure 50 shows various types of orifices of the same diameter. The first (a) is the above mentioned standard orifice. The second (b) is a clean-cut hole in a plate of measurable thickness. It has the same coefficients as (a) provided its thickness is small. The third (c) is the reverse of (a) and, while it forms a contracted jet, its coefficient of discharge is

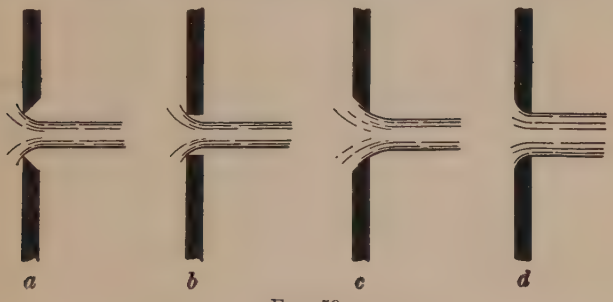


FIG. 50

much greater and depends on the shape of the bevels. The last (d) shows the inner edge carefully rounded to conform to the shape of the contracted vein, and the probable coefficient is only a little less than unity, as the contraction coefficient is itself unity.

It is evident that it is essential to use only standard orifices when endeavoring to accurately measure discharge, unless the value of c for the particular orifice has been predetermined.

48. Large Vertical Orifices under Low Heads.—So far we have dealt with an orifice whose vertical dimension has been small compared to the head upon it. With large orifices under low heads, the variation of velocity in the jet's cross-section gives rise to a discharge differing slightly from that obtained by use of formula (37). It will now be shown, however, that the difference is slight and may be neglected, provided the head is at least twice the vertical dimension of the orifice.

Case 1. *Circle* (Fig. 51).—As before, h will represent the head on the center of the orifice. If A-B be any elementary strip, drawn horizontally across the orifice at a distance x from its center, we have for a small discharge through it:—

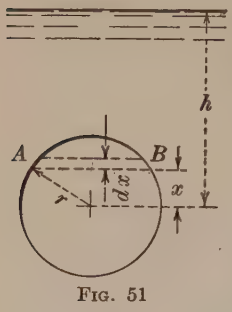


FIG. 51

$$dQ = dA \,.\, v = 2\sqrt{r^2 - x^2}\,.\,dx \times \sqrt{2g\,(h - x)}$$
$$= 2\sqrt{2g}\,(r^2 - x^2)^{\frac{1}{2}}(h - x)^{\frac{1}{2}}\,dx.$$

By making x vary between the values $-r$ and $+r$, the integration of this expression will give the discharge from the entire orifice. It will be necessary to expand the term $(h - x)^{\frac{1}{2}}$ by the binomial theorem.

$$(h - x)^{\frac{1}{2}} = h^{\frac{1}{2}} - \frac{h^{-\frac{1}{2}}x}{2} - \frac{h^{-\frac{3}{2}}x^2}{8} - \frac{h^{-\frac{5}{2}}x^3}{16} - \text{etc. . . .}$$

$$\therefore dQ = 2\sqrt{2g}\left[(r^2 - x^2)^{\frac{1}{2}}h^{\frac{1}{2}} - \frac{(r^2 - x^2)^{\frac{1}{2}}x}{2\,h^{\frac{1}{2}}} - \frac{(r^2 - x^2)^{\frac{1}{2}}x^2}{8\,h^{\frac{3}{2}}} - \right.$$
$$\left. \frac{(r^2 - x^2)^{\frac{1}{2}}x^3}{16\,h^{\frac{5}{2}}} - \text{etc. . . .}\right] dx.$$

Each term of this is now possible of integration, and there results

$$Q = \pi r^2\sqrt{2\,gh}\,(1 - \frac{r^2}{32h^2} - \frac{5r^4}{1024h^4} - \frac{105r^6}{65537h^6} - \text{etc.}) \quad . \quad (39)$$

which is an exact theoretic formula for the discharge. An inspection of the parenthesis quantity shows it to have a value less than unity, and the discharge is therefore less than that given by the formula

$$Q = a\sqrt{2\,gh}$$

previously obtained for relatively large heads. This is what we might have anticipated.

Let us inquire into the numerical value of the parenthesis quantity as we assign different values to the ratio $\frac{h}{r}$. If $\frac{h}{r} = 2$, or the head is just equal to the diameter, the quantity becomes 0.992. With $\frac{h}{r} = 4$, or $h = 2\,d$, the value is 0.998. The latter figure

shows that with all ordinary ranges of head we may neglect the refinement of formula (39), and for figuring actual discharge use

$$Q = ca\sqrt{2gh},$$

the value of c being determined by aid of Art. 46. If Hamilton Smith's values be selected it will be necessary to use them in connection with equation (39) whenever the value found in the table lies above the horizontal line drawn across the several columns of the table.

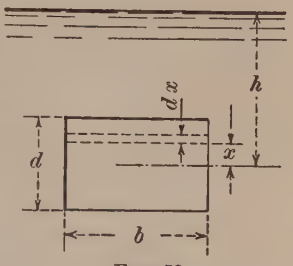


FIG. 52

Case 2. *Rectangle* (Fig. 52).—In this case the small discharge dQ through an elementary strip parallel to the surface may be written, as before,

$$dQ = dA \,.\, v = b \,.\, dx \,.\, \sqrt{2\,g(h - x)}$$

or

$$dQ = b\sqrt{2\,g}(h - x)^{\frac{1}{2}} \,.\, dx$$

where the limits of x are $-\frac{d}{2}$ and $+\frac{d}{2}$. If we expand $(h - x)^{\frac{1}{2}}$ as before,

$$Q = b\sqrt{2\,gh}\int_{-\frac{d}{2}}^{\frac{d}{2}}\left(1 - \frac{x}{2\,h} - \frac{x^2}{8\,h^2} - \frac{x^3}{16\,h^3} - \frac{5x^4}{128\,h^4} - \text{etc.}\ .\ .\right)dx,$$

or

$$Q = bd\sqrt{2\,gh}\left(1 - \frac{d^2}{96\,h^2} - \frac{1}{2048}\frac{d^4}{h^4} - \text{etc.})\,.\right). \quad . \quad . \quad . \quad (40)$$

As in the previous case, we see that the value of the parenthesis is less than unity. If $h = d$, its value becomes 0.989, while for $h = 2\,d$, it becomes 0.997. Then we may say that for heads greater than twice the depth of the orifice, we may figure the actual discharge from

$$Q = ca\sqrt{2\,gh}.$$

The following tables furnish values of c for square orifices and for rectangular orifices 1 ft. wide and of varying depths.

Coefficients of Discharge (c) for Square Orifices

From Hamilton Smith's "Hydraulics"

Head h in Feet	Side of the Square in Feet						
	0.02	0.04	0.07	0.1	0.2	0.6	1.0
0.4		0.643	0.628	0.621			
0.6	0.660	.636	.623	.617	0.605	0.598	
0.8	.652	.631	.620	.615	.605	.600	0.597
1.0	.648	.628	.618	.613	.605	.601	.599
1.5	.641	.622	.614	.610	.605	.602	.601
2.0	.637	.619	.612	.608	.605	.604	.602
2.5	.634	.617	.610	.607	.605	.604	.602
3.0	.632	.616	.609	.607	.605	.604	.603
4.0	.628	.614	.608	.606	.605	.603	.602
6.0	.623	.612	.607	.605	.604	.603	.602
8.0	.619	.610	.606	.605	.604	.603	.602
10.0	.616	.608	.605	.604	.603	.602	.601
20.0	.606	.604	.602	.602	.602	.601	.600
50.0	.602	.601	.601	.600	.600	.599	.599
100.0	.599	.598	.598	.598	.598	.598	.598

Coefficients of Discharge (c) for Rectangular Orifices 1 Foot Wide

From Fanning's Treatise on Hydraulic and Water Supply Engineering

Head h in Feet	Depth of Orifice in Feet						
	2.0	1.5	1.0	0.75	0.50	0.25	0.125
0.4						0.619	0.625
0.6					0.614	.618	.623
0.8				0.609	.612	.616	.622
1.0		0.608	0.605	.608	.611	.616	.622
1.5	0.609	.607	.604	.607	.610	.614	.620
2.0	.609	.606	.604	.606	.609	.613	.619
2.5	.608	.606	.603	.606	.609	.613	.617
3.0	.607	.605	.603	.605	.608	.612	.616
4.0	.606	.604	.602	.604	.606	.610	.614
6.0	.604	.603	.601	.602	.604	.608	.610
8.0	.603	.602	.601	.601	.603	.605	.608
10.0	.601	.601	.601	.601	.602	.604	.606
15.0	.602	.601	.601	.601	.601	.603	.607
20.0	.602	.601	.601	.601	.602	.604	.607

Some experiments by Ellis on a large orifice 2 ft. square under heads varying from 1.8 to 11.3 feet gave a coefficient nearly constant in value and equal to 0.60. This would apparently indicate the existence of a *critical head* as explained in Art. 46. Another and simpler form of equation than (40) may be found in Art. 70.

49. Recapitulation.—It is well to bear in mind the conditions under which we have so far studied orifice flow. We have assumed:—

(*a*) No suppression of the contraction.

(*b*) Ratio of reservoir surface to orifice area, very large, *i. e., no appreciable velocity of approach.*

(*c*) Reservoir surface and jet both under atmospheric or the *same* pressure.

A departure from any one of these conditions will lead to material changes in the flow, the nature of which will be shown in the succeeding paragraphs.

50. Suppression of the Contraction.—The location of the orifice with respect to its distance from the sides and bottom of the

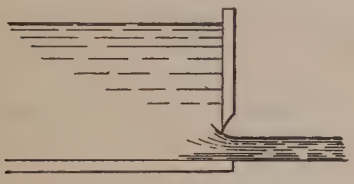


Fig. 53

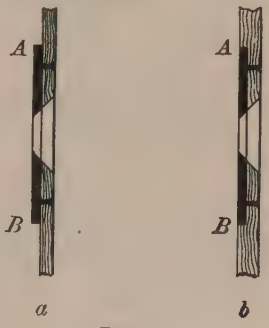


Fig. 54

reservoir is a matter of importance. If it be placed so that its edge be flush with any one side, as in Fig. 53, the contraction of the jet on that side of the orifice will be wholly suppressed. Experiment has shown that the contraction is not fully restored and made "*complete*" until the orifice is moved far enough away from the side to provide a free lateral approach from all directions for a distance equal to *three times the least dimension of the orifice.* If placed nearer, the partial suppression of the contraction results in a change in the value of the coefficient of discharge.

Orifices are sometimes improperly constructed as in Fig. 54*a*

and the plate A-B prevents the lateral approach. A preferable construction would be that shown in Fig. 54*b*. If the orifice were made in the side of a cylindrical tank, the curvature of the tank might slightly diminish the contraction.

51. Velocity of Approach.—Figure 55 shows, in longitudinal section, a channel bringing water to an orifice. Because of a relatively small cross-section, there exists in the channel a velocity of approach which we will *assume* to have a value v_o at all points in the section (not really so). Applying our theorem to the points m and n, we obtain

$$\frac{v_o^2}{2g} + 0 + h = \frac{v^2}{2g} + 0 + 0$$

or

$$v = \sqrt{2g\left(h + \frac{v_o^2}{2g}\right)} \quad . \quad . \quad . \quad . \quad . \quad (41)$$

this representing the *ideal* value of v. Since in general, v_o is not known numerically, another equation between v and v_o must be obtained. This is furnished by the equation of continuity of flow, expressing the fact that the quantity passing the cross-section of the channel (of area A) and the contracted section of the jet (area $c_c a$) is the same.

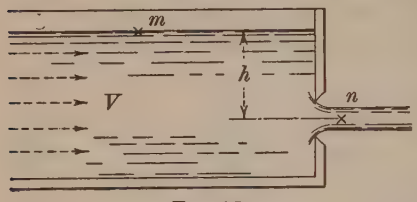


FIG. 55

Thus,

$$Av_o = (c_c a)\ (c_v v)$$

where $c_v v$ represents the *real* velocity at the contracted section. Substituting the value of v_o obtained from this equation in (41) and simplifying

$$v = \sqrt{\frac{2gh}{1 - \left(\frac{ac}{A}\right)^2}}.$$

$\therefore$ for the actual velocity,

$$v = c_v \sqrt{\frac{2gh}{1 - \left(\frac{ac}{A}\right)^2}}, \quad . \quad . \quad . \quad . \quad . \quad (42)$$

and

$$Q = ca \sqrt{\frac{2gh}{1 - \left(\frac{ac}{A}\right)^2}} \quad . \quad . \quad . \quad . \quad . \quad (43)$$

The equation does not contain v_o, hence it is convenient to use when v_o is unknown and the area, A, given.

It will be found that the velocity of approach becomes negligible when

$$A \geqq 15a.$$

In the previous article it was pointed out that a standard orifice must be placed at least three times its least dimension from the sides of the reservoir in order that its coefficient be unaffected by suppression of contraction. This would result in

$$A = 50a, \text{ approximately.}$$

Evidently a standard orifice, properly constructed is not exposed to a velocity of approach which need be considered.

Formulas (42) and (43) were derived under the assumption that v_o was uniform in its value at all parts of the cross-section. This is not so, but the error involved cannot be larger than that arising from an estimate of the orifice coefficient, since the latter will be affected by the partial suppression of the contraction which exists whenever the sides of the channel are close enough to the orifice to produce a considerable velocity of approach.

Evidently, although the total head is $h + \frac{v_o^2}{2g}$, the coefficient may be estimated by a consideration of h only.

Example.—For the conditions shown in Fig. 55, assume the following data:—

$h = 9$ inches. $A = 1$ sq. ft. Diam. of orifice, 6 inches.

Compute the probable rate of discharge, assuming that conditions warrant assuming c equal to 0.70.

Using equation (43),

$$Q = 0.196 \sqrt{\frac{64.4 \times .75}{\frac{1}{.49} - \left(\frac{.196}{1}\right)^2}} = 0.96 \text{ cu. ft. per sec.}$$

If the velocity of approach had been neglected the computed discharge would have been about one per cent less.

52. Flow under Pressure.—If the intensity of pressure on the water surface in the reservoir and that surrounding the jet are not equal, the formula, $Q = ca\sqrt{2gh}$, does not apply if h represents the

static head over the orifice. If h be made to represent the sum of static head and the difference of pressure-head at these two named points, the equation still holds provided there be no velocity of approach. In any case it is probably best to proceed with writing our general equation between a point in the reservoir surface and one in the contracted section of the jet.

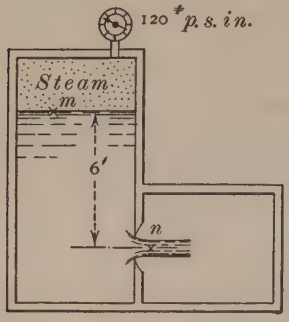


Fig. 56

Example.—What will be the velocity of flow and rate of discharge from the standard orifice in Fig. 56 when steam under a gauge pressure of 120 lb. per square inch fills the space above the water, and the receiving tank has in it a pressure of 4 lb. less than the atmospheric? Diameter of orifice is 2 inches.

Bernoulli's Equation for points m and n gives

$$0 + \frac{120 \times 144}{62.5} + 6 = \frac{v^2}{64.4} - \frac{4 \times 144}{62.5} + 0$$

Ideal $v = 137$ ft. p. s.

Actual $v = .98 \times 137 = 134$ ft. p. s.

From the first equation it will be found that the velocity is due to a head of 291.2 ft., and using Bilton's values of c (Art. 46), the coefficient of discharge is 0.599. Smith's value is 0.592.

$Q = .599 \times .022 \times 137 = 1.81$ cu. ft. p. s.

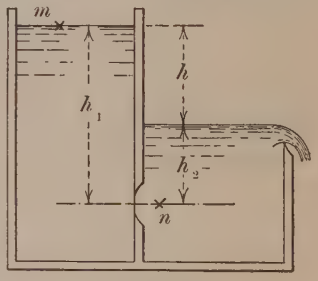


Fig. 57

53. Submerged Orifice.—If an orifice discharges wholly under water, it is said to be *Submerged.* That the theoretic velocity equals $\sqrt{2gh}$, where h is the difference in the water levels, can be easily shown, assuming atmospheric pressure to be on the two surfaces. For the points m and n we have (Fig. 57),

$$0 + 34 + h_1 = \frac{v^2}{2g} + (34 + h_2) + 0,$$

from which

$$v^2 = 2g(h_1 - h_2),$$

or $$v = \sqrt{2gh}.$$

As before, $$Q = ca\sqrt{2gh},$$

but the values of c are now different from those given for the case of an orifice discharging into air. Experiment indicates a slight decrease in the value of the coefficient, as may be noted in the following table based on Smith's experiments.

Submerged orifices are common in engineering works, being found in locks, waste-ways, tide gates, and many other constructions.

COEFFICIENTS OF DISCHARGE (c) FOR SUBMERGED ORIFICES

Based on Data from Hamilton Smith's "Hydraulics"

Effective Head in Feet	Size of Orifice in Feet				
	Circle **0.05**	Square **0.05**	Circle **0.1**	Square **0.1**	Rectangle **0.05 × 0.3**
0.5	0.615	0.619	0.603	0.608	0.623
1.0	.610	.614	.602	.606	.622
1.5	.607	.612	.600	.605	.621
2.0	.605	.610	.599	.604	.620
2.5	.603	.608	.598	.604	.619
3.0	.602	.607	.598	.604	.618
4.0	.601	.606	.598	.604	

54. Water Measurement by Orifices.—Next to volumetric measurements or weighing, the orifice furnishes the most accurate way of measuring moderate rates of discharge. With standard orifices and precautions to secure complete contraction and a negligible velocity of approach, the result should not be in error more than about 1 per cent. Care is necessary in measuring the head, and for this purpose either a piezometer tube or a hook gauge may be used. If the head exceeds 2 feet, the piezometer will be found to furnish all the necessary accuracy, as the head may be easily read to one one-hundredth of a foot; and an error of this amount in reading the gauge at this head would result in an error in the figured discharge of less than one-third of 1 per cent. As commonly used the tube is of glass connected with the tank at a point somewhat below the surface level by means of metal or rubber connections, and carried vertically upward to a point above the surface. On its sides, or attached to it, is a scale divided into feet, tenths, and hundredths of feet, with its zero at the orifice's center

The position of the lowest point of the curved meniscus, forming the top of the water column, is then easily read.

55. The Hook Gauge.—For low heads, or when great accuracy is required, use is made of the hook gauge in determining the height of the surface level. This may be briefly described as a graduated metal rod arranged to move vertically between fixed supports and having at its lower end a pointed hook. The rod is controlled in its vertical movement by a slow motion screw, and a vernier attachment enables its position to be read to the one-thousandth part of a foot. The accompanying illustration shows a gauge as manufactured by W. and L. E. Gurley of Troy, N. Y.

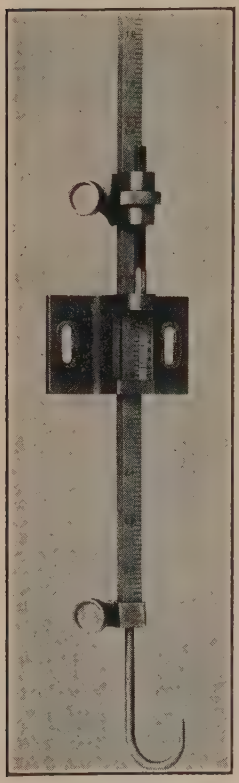


Fig. 58

In using the gauge, the hook is lowered beneath the surface of the water and then raised until the point causes a tiny pimple or elevation to appear on the surface. This latter is really a distortion of the "skin" which is formed on the water by surface tension. The hook is then lowered sufficiently to cause this to be barely perceptible, and the point of the hook is assumed to be in the water's surface. The vernier is then read. The position of the hook when at the orifice's center must be carefully determined by previous measurement, and the head on the orifice is evidently the difference between the surface reading and this initial or zero reading.

The hook gauge is well adapted to measuring differences in water level and is much used in experimental work. By it, differences in level of 1/50 of an inch are readily detectable and with a little practice a level may be determined with an error of about 1/200 of an inch.

56. The Miner's Inch.—Mention must be made, in passing, of the manner of measuring water which has for years been in vogue

in the western part of the United States. In the mining section water is commonly sold by the miner's inch, which may be approximately defined as the amount of water which will pass in 24 hours a sharp-edged orifice, one inch square, under a head of $6\frac{1}{2}$ inches. Inasmuch as this is a relatively small quantity, it is customary to measure the water through orifices of much larger size and assume that as many inches are delivered as there are square inches in the orifice's area, the head being kept at 6 inches on the *upper edge*. This, with other practices connected with the mode of measurement, makes the miner's inch a variable and vague quantity; and it is to be hoped that gradually this unsatisfactory and unscientific method will be discarded and water sold by the cubic foot.

57. Discharge under a Falling Head.—It is often required to find the time necessary to empty a reservoir or draw down its level a certain amount. Let h_1 (Fig. 59) be the head on the orifice at the moment of opening and h_2 the head at the end of some interval of time t. We will assume the horizontal cross-section of the reservoir to be constant. At any particular instant the velocity of flow is

$$v = c_v \sqrt{2gh},$$

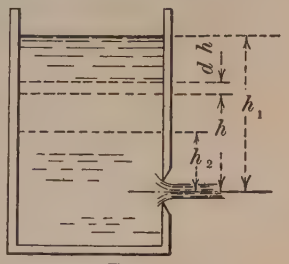


FIG. 59

where h is the head at that instant. The quantity that would be discharged per second under that head is

$$Q = ca\sqrt{2gh},$$

and in a very small fraction of time dt a small quantity

$$dQ = ca\sqrt{2gh}\,dt$$

would be discharged. In the same interval of time the head would drop a small amount dh, and the amount passing from the reservoir can be expressed as the drop times the cross-sectional area A, *i. e.*

$$dQ = dh \times A.$$

Equating these values of dQ,

$$dhA = ca\sqrt{2gh}\,dt,$$

or $$dt = -\frac{A\,dh}{ca\sqrt{2gh}},$$

the minus sign expressing the fact that $\frac{dt}{dh}$ is negative since dt is an *increment* and dh a decrement. Since h varies from h_2 to h_1, the integration of the above will give

$$t = -\frac{A}{ca\sqrt{2g}}\int_{h_1}^{h_2} h^{-\frac{1}{2}}dh$$

or $$t = \frac{2A}{ca\sqrt{2g}}(h_1^{\frac{1}{2}} - h_2^{\frac{1}{2}}). \quad . \quad . \quad . \quad . \quad (44)$$

If the horizontal cross-section of the reservoir is not constant in area, the problem is still possible of solution if A be expressed in terms of h. Where, as above, the surface area of the reservoir remains *constant*, we may determine the average value of the velocity of flow from the orifice as follows:

$$v_m = \frac{\text{Total } Q}{\text{Area of jet} \times \text{time}} = \frac{A(h_1 - h_2)}{c_c a \times \frac{2A(\sqrt{h_1} - \sqrt{h_2})}{c_d a\sqrt{2g}}},$$

or $$v_m = c_v\sqrt{2g}\,\frac{(h_1^{\frac{1}{2}} + h_2^{\frac{1}{2}})}{2} = \frac{c_v\sqrt{2gh_1} + c_v\sqrt{2gh_2}}{2}. \quad . \quad (45)$$

Thus we find that the *average* velocity during the time of discharge is the arithmetical mean of the velocity due to h_1 and that due to h_2.

Since

Total Volume Discharged = Average rate $\times$ time, we have

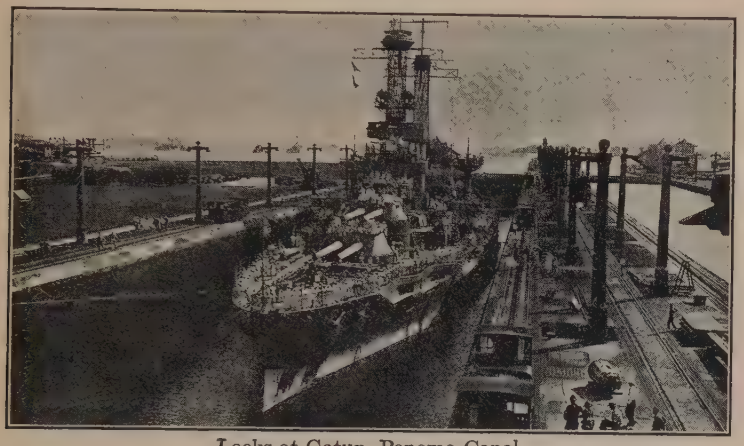
$$A(h_1 - h_2) = c_c a v_m \times t,$$

and $$t = \frac{2A(h_1 - h_2)}{ca(\sqrt{2gh_1} + \sqrt{2gh_2})}. \quad . \quad . \quad . \quad . \quad (46)$$

It is evident that (44) and (46) differ only in the arrangement of terms, but the use of the average rate of discharge makes the solution of problems very easy. Again it should be noted that only with A constant can the average velocity method be used.

Example.—One of the locks on the Lachine Canal has a surface area of 12,150 sq. ft., and its water level is 9 feet below that

of the water in the canal above the lock. The gate between the two levels is supplied with two sluices and the level of the water in the lock is raised to that above the gate in 2 minutes and 48 sec-



Locks at Gatun, Panama Canal

onds. Assuming c equal to 0.60 and no fluctuation in the canal level, determine the area of each sluice opening.

Since the head varies from 9 ft. to zero, the average rate of discharge is

$$Q_m = ca\left(\frac{\sqrt{2gh_1} + \sqrt{2g\,h_2}}{2}\right)$$

$$= 0.6a \times \frac{\sqrt{64.4 \times 9}}{2}$$

$$= 0.6a \times 12.04 = 7.22a$$

Total Volume Discharged $= 12150 \times 9 = 109350$ cu. ft.

$$t = 168 \text{ sec.} = \frac{109350}{7.22a}$$

$$a = 90 \text{ sq. ft.}$$

As this is the combined area of two sluices, each must have an area of 45 sq. ft. *Ans.*

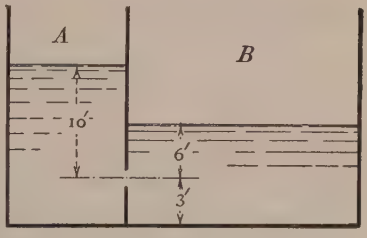


Fig. 60

Example.—A prismatic vessel (Fig. 60) has two compartments A and B, communicating by a standard orifice 6 inches square, its center being 3 feet above the bottom of the vessel. The

horizontal cross-section of A is 100 sq. ft., and that of B is 200 sq. ft. At a certain time the water stands 13 ft. deep in A and 9 ft. deep in B. How soon thereafter will the surfaces reach a common level? Assume $c = .60$.

Let y be the depth of water in A at any instant, and dy be the change in depth during any interval of time dt. The rise in B's level during the same time will be dy $(100 \div 200)$, and the net change, dh, in head will be

$$dh = \tfrac{3}{2}\, dy.$$

Through the orifice there flows in dt seconds

$$dQ = ca\sqrt{2gh}\, dt;$$

also
$$dQ = 100\, dy = \tfrac{200}{3}\, dh.$$

$$\therefore -\tfrac{200}{3}\, dh = ca\sqrt{2gh}\, dt = 0.6 \times \tfrac{36}{144} \times 8.02 \times h^{\frac{1}{2}}\, dt.$$

$$t = -55.4 \int_{4}^{c} h^{-\frac{1}{2}} dh.$$

$$t = 221.6 \text{ sec.} \quad Ans.$$

Solution by using average rate of discharge.—

$$Q_m = .6 \times \frac{36}{144}\sqrt{\frac{64.4 \times 4}{2}} = 1.2 \text{ c. f. p. s.}$$

Level in A descends x ft. Therefore B's level rises $\frac{x}{2}$ ft.

$$x + \frac{x}{2} = 4$$

$$x = 2\tfrac{2}{3} \text{ ft.}$$

Total Volume leaving $A = 100 \times 2\tfrac{2}{3} = 267$ cu. ft.

$$t = \frac{267}{1.2} = 221.6 \text{ sec.} \quad Ans.$$

58. Discharge under Falling Head with Constant Inflow to Reservoir.—Let the rate of inflow be denoted by Q_o. Starting with any head, h, as in Fig. 59, the drop in dt seconds would be dh and out from the reservoir would flow a quantity, dQ, which would be expressed as $A dh$ *plus the amount* $Q_o dt$ *which would enter in the same time.*

Proceeding as in Art. 57,

$$-A\,dh + Q_o dt = ca\sqrt{2gh}\,dt$$

$$dt = -\frac{A\,dh}{ca\sqrt{2gh} - Q_o}$$

the minus sign in the right-hand member expressing the fact that $\frac{dt}{dh}$ is negative as dt is an increment and dh a decrement.

Locks with Six Lifts on the Rideau Canal, Ottawa

$$t = \int_o^t dt = -A\int_{h_1}^{h_2} \frac{dh}{ca\sqrt{2gh} - Q_o}$$

$$t = \frac{A}{c^2a^2g}\left[ca\sqrt{2g}\,(\sqrt{h_1} - \sqrt{h_2}) + Q_o\,log_e\,\frac{ca\sqrt{2gh_2} - Q_o}{ca\sqrt{2gh_1} - Q_o}\right]$$

59. Head Lost at an Orifice.—To ascertain how much head is lost by friction as water passes an orifice, we may proceed as follows:

At the contracted section,

$$v = c_v\sqrt{2gh}$$

and the velocity head at the same point is

$$\frac{v^2}{2g} = c_v^2 h.$$

Had no loss by friction taken place, the velocity head in the jet would have been h; consequently

$$\text{Lost Head} = h - c_v^2 h = (1 - c_v^2)\, h. \quad . \quad . \quad . \quad (47)$$

A more convenient form for use in Bernoulli's Equation may be obtained by replacing h with its value, $\frac{1}{c_v^2} \cdot \frac{v^2}{2g}$ obtained from the second equation.

$$\text{Lost Head} = \left(\frac{1}{c_v^2} - 1\right)\frac{v^2}{2g} \quad . \quad . \quad . \quad . \quad (48)$$

For a standard orifice ($c_v = 0.98$),

$$\text{Lost Head} = 0.04\, h, \text{ or } 0.041\, \frac{v^2}{2g}.$$

Expression (48) is more often used than (47), inasmuch as it contains the velocity head and, when inserted in Bernoulli's Equation, permits an easy solution of the equation for the *real* velocity. Both equations (47) and (48) are applicable, as will later be seen, to any discharging device whose coefficient of velocity is known.

Example.—Solve the example given in Art. 52 by using Bernoulli's Equation with lost head inserted. From conditions as given,

$$0 + \frac{120 \times 144}{62.5} + 6 = \frac{v^2}{64.4} - \frac{4 \times 144}{62.5} + \left(\frac{1}{(0.98)^2} - 1\right)\frac{v^2}{64.4}$$

$$v = 134 \text{ ft. per second.}$$

PROBLEMS

1. Compute the rate of discharge in gallons per minute from a standard 2-inch circular orifice, using

(*a*) Smith's coefficient;

(*b*) Judd and King's coefficient;

(*c*) Barnes' formula.

2. A circular orifice, 8 inches in diameter and having a thin edge, is made in the side of a reservoir which receives water at the rate of 6 cu. ft. per second. Determine the height above the center of the orifice to which water will rise in the reservoir.

(*a*) Use Smith's coefficient;

(*b*) Use Barnes' formula.

3. Water spurts vertically upward from an orifice in a horizontal plane under a head of 40 ft. What is the limit to the height of the jet if the coefficient of velocity be 0.97 and air friction negligible?

Ans. 37.6 ft.

4. A fluid of one fourth the density of water is discharged into the free atmosphere from an orifice in the side of a reservoir. If the pressure at a point in the reservoir on the same horizontal plane as the orifice be 50 lb. per square inch (absolute units), find the theoretic velocity of discharge.

Ans. 144.2 ft. per second.

5. Find the theoretic discharge through a vertical, sharp-edged orifice having the form of an isosceles triangle with an altitude of 10 inches, base (horizontal) 6 inches, and vertex level with the reservoir's surface. Disregard the velocity of approach.

Ans. 1.22 cu. ft. per second.

6. Water spurts out horizontally from a half-inch round hole in the side of a timber penstock, the wall of which is 2 in. thick. The jet strikes at a point 6 ft. horizontally distant from the orifice and 2 ft. lower down. Assuming the stream to issue from the orifice without contraction, due to the thickness of the wood penstock, compute the probable leakage in gallons per 24 hours.

Ans. 14,960 gal.

7. The jet from a circular, sharp-edged orifice, ½ inch in diameter, under a head of 18 feet strikes a point distant 5 feet horizontally and 4.67 inches vertically from the center of the orifice. The discharge is 118.95 gallons in 569.2 seconds. Compute the coefficient of discharge, velocity, and contraction.

Ans. $C_d = 0.60.$
$C_v = 0.945.$
$C_c = 0.635.$

8. A steel box, rectangular in plan, floats with a draft of 2 ft. If the box be 20 ft. long, 10 ft. wide, and 6 ft. deep, compute the time necessary to sink it to its top edge by opening a standard orifice, 6 in. square, in its bottom. Neglect the thickness of the vertical sides.

Ans. 472 sec.

9. What will be the rate of discharge through a 1-inch orifice in the bottom of a vessel moving upward with an acceleration of 10 ft. per second each second, the water being 8 ft. deep over the orifice? Assume $C_d = 0.61.$

Ans. 0.087 cu. ft. per second.

10. A reservoir has a side wall, inclined backward 30 degrees from the vertical, in which is an orifice having a velocity coefficient of 0.97. With a head of 9 ft. on the orifice, compute

(*a*) Vertical height, above center of orifice, which jet will reach.

(*b*) Horizontal distance from orifice to jet.

(*c*) Velocity of jet as it strikes the ground a distance 7 ft. below orifice.

Ans. (*a*) 2.12 ft.
(*b*) 14.7 ft.
(*c*) 31.6 ft.

11. A standard 2-inch orifice connects two vessels, A and B. In A the water stands 4.6 ft. deep above the center of the orifice and its surface is exposed to steam at a pressure above atmospheric equivalent to 18 in. of mercury. In B the level of the water is 1 ft. above the orifice center and the air pressure on it is less than atmospheric by an amount corresponding to 10 in. of mercury. Compute the probable discharge in cubic feet per second.

Ans. 0.62 cu. ft. per second.

12. A sharp-edged orifice, 1 ft. square, discharges under a head of 1 ft., the latter being measured above the center of the orifice. Compute the rate of discharge assuming the mean head to be 1 ft., and compare the result with that obtained by using the exact formula in paragraph 48.

Ans. 4.81 cu. ft. per second.
4.75 cu. ft. per second.

13. A reservoir of water is connected with one containing oil by a sharp-edged circular orifice 2 in. in diameter. Determine the rate of discharge when the water stands 6 ft., and the oil 2 ft., deep on the orifice's center. The oil has a specific gravity of 0.80.

Ans. 0.22 cu. ft. per second.

14. A thin plate closes the end of a 6-inch pipe and a 3-inch orifice is made in the center of the plate. At what rate will water be discharged through the orifice when a pressure of 10 lb. per sq. in. is maintained in the pipe a short distance back from the orifice? Assume $C_v = 0.98$, $C_c = 0.61$, and make the computations with and without consideration of the effect of velocity of approach.

15. Find the head necessary in order that the energy of the jet from a 2-inch orifice shall be 2 horsepower. Assume $C_c = 0.65$ and $C_v = 0.96$.

$h = 31.5$ ft.

16. A sharp-edged orifice of 4 sq. in. cross-section (plane vertical) discharges a jet of water which at a point 3 ft. below the center of the con-

tracted section is 4 ft. in front of it. Compute the probable discharge in gallons per hour. $C_v = 0.98$ and $C = 0.608$.

17. A hollow cone, 4 ft. in diameter and 6 ft. high is placed in a vertical position, with vertex down, and filled with water. If while rotating about its axis at a rate of 60 revolutions per minute, a small orifice at the vertex be opened, with what velocity will the water escape during the first short interval of time? Assume $C_v = 0.99$.

18. A vessel 16 ft. high and 6 ft. in diameter contains water 7 ft. deep when at rest. How fast must it be revolved about a vertical axis through its center in order that the velocity (relative to the vessel) of flow from an orifice in the side and 1 ft. above the bottom, may be 24 ft. per second?

Ans. 65.8 R. P. M.

19. A cylindrical vessel, 10 ft. high and 4 ft. in diameter, is filled with water to a depth of 8 ft. In its side is a 2-inch circular orifice ($c_d = 0.60$) placed 1 ft. above the bottom of the vessel. If the vessel be rotated about its own axis at 60 R. P. M. and the water allowed to escape through the orifice for a period of 3 minutes, how deep will the water stand in the vessel if brought to rest?

20. A rectangular tank with vertical sides, 16 ft. long and 4 ft. wide, contains water to a depth of 4 ft. How long will it take to empty it by opening a 4-inch, sharp-edged, circular orifice in its bottom, assuming a constant discharge coefficient of 0.60?

Ans. 610 sec.

21. The head in a vessel with vertical sides, at the instant of opening an orifice, was 9 ft. and at closing had decreased to 5 ft. Determine the constant head under which in the same time the orifice would discharge the same volume of water. *Ans.* 6.85 ft.

22. From a given prismatic reservoir the discharge from an orifice at the base would be 32 cu. ft. in t seconds if the head were constant at 16 ft.

(*a*) Under a falling head, starting at 25 ft., how much must it fall during the same time, t, so that the discharge during that time shall also be 32 cu. ft., assuming the coefficient of discharge constant?

(*b*) What is the area of the vessel in square feet?

(*c*) If $c_d = 0.85$ and $t = 40$ seconds, what would be the area of the orifice?

Ans. Fall = 16 ft.
Area = 2 sq. ft.
a = .029 sq. ft.

23. A prismatic vessel has two compartments, A and B, communicating by a standard orifice 12 inches square with a coefficient, C, of 0.60. The horizontal cross-section of A is 100 sq. ft. and that of B is 400 sq. ft. At a certain time the water in A is 18 ft. above the center of the orifice, and in B 9 ft. How soon thereafter will the water surfaces be 4 ft. apart?

24. A reservoir is being emptied by means of an orifice in its bottom which has an area of 10 sq. ft. and a coefficient, c, of 0.80. In plan the reservoir is circular, but its horizontal section varies in diameter as is shown by the relation, $d = \frac{h^2}{\sqrt{\pi}}$, h being the head in the orifice. When full, the reservoir has an area of 40,000 sq. ft. How long a time will be required to empty it?

25. A canal lock 400 ft. long and 90 ft. wide is emptied through a submerged opening having an area of 60 sq. ft. and a discharge coefficient of 0.64. The water discharges into the lower canal, which is maintained at a constant level 20 ft. below that in the lock when full. How long will it take to bring the lock level down to that in the lower canal?

Ans. 17.4 min.

26. Water fills a vessel whose shape is that of an inverted pyramid with a square base. If the base measures 12 inches on a side and the altitude is 2 feet, how long a time will be required to empty the vessel through an orifice at the apex 1 inch square. Assume that $c_d = 0.88$ and disregard changes in dimensions of pyramid caused by cutting the orifice.

Ans. 11.6 sec.

27. A reservoir, $\frac{1}{2}$ acre in area (21,780 sq. ft.), with sides nearly vertical so that it may be considered prismatic, receives a stream yielding 9 cu. ft. per second, and discharges through a sluice 4 ft. wide which is raised 2 ft. Calculate the time necessary to lower the surface 5 ft., the head over the center of the sluice, when opened, being 10 ft. Assume $c_d = 0.62$.

Ans. 1145 sec.

28. If the actual velocity of flow from a certain orifice under an 8-foot head was found to be 22.09 ft. per second, what was the loss in head by friction?

Ans. 0.43 ft.

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CHAPTER VI

FLOW THROUGH MOUTHPIECES

60. General.—If in any manner a mouthpiece be added to an orifice, we may expect changes in both velocity and quantity discharged. As for velocity, we notice a general diminution on account of increased frictional resistances, eddyings, impact, etc. The quantity discharged is either increased or diminished, according to the construction of the mouthpiece.

61. Standard Short Tube.—If a short cylindrical tube, having a length of 2 to $2\frac{1}{2}$ diameters, be attached to the orifice on its outer side, we have what is known as the standard short tube (Fig. 61). If the flow be started by first stopping the tube at exit, and then allowing the water to escape, the jet will issue in parallel filaments, suffering no contraction, and completely filling the tube at the end. Inside the tube at m the jet is contracted by passing the sharp edges of the opening, but quickly expands to fill the tube. By reason of the eddying, which follows the expansion, considerable friction is developed. This, added to friction at entrance and along the sides of the tube, tends to materially reduce the velocity. If Bernoulli's Theorem be applied between a point in the reservoir surface and one in the free jet, we obtain, as we did for the orifice,

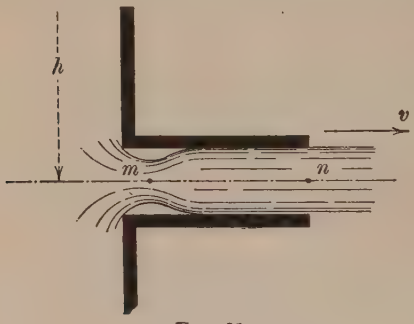

FIG. 61

$$v = \sqrt{2gh}.$$

Weisbach obtained from experiment a mean value of

$$v = 0.815\sqrt{2gh}, \quad . \quad . \quad . \quad . \quad . \quad (49)$$

the coefficient varying slightly with the head and diameter of tube. We may then write

$$Q = 0.82a\sqrt{2gh},$$

inasmuch as the contraction coefficient is unity. Thus we have a discharge which is one third greater than that obtained from the standard orifice, although the *velocity* of the discharge is much less.

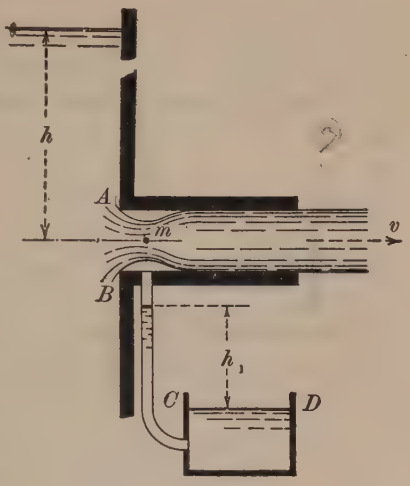

FIG. 62

If a tube be tapped into the mouthpiece at the point where the contraction takes place, and its lower end be immersed in water (Fig. 62), the latter will rise in the tube, showing that the pressure in the contracted section is less than atmospheric. It will be interesting to determine the relation between the height of the water column h_1 and the static head h on the mouthpiece.

Between a point in the reservoir surface and one at m we may write

$$0 + \frac{p_a}{w} + h = \frac{v_m^2}{2g} + \frac{p_m}{w} + 0 + \text{Lost head},$$

and if v_m and the lost head can be expressed in terms of h, this general equation may be solved for $\frac{p_m}{w}$ in terms of h. The value of $\frac{v_m^2}{2g}$ may be found by assuming a coefficient of contraction for the stream at m. In the case of a standard orifice this is about 0.62; but there is reason to believe that here the contraction is slightly less, due to less pressure on the section, and possible backing up of the water beyond. We will assume $c_c = 0.64$. Then, with steady flow,

$$v_m \times 0.64\, a = v \times a,$$

or $$v_m = 1.56\, v = 1.56 \times 0.82\sqrt{2gh} = 1.28\sqrt{2gh},$$

from which $$\frac{v_m^2}{2g} = 1.64\ h.$$

We have next to determine the value of the head lost in friction as the water enters the tube past the edge *A-B*, this being the only sensible loss occurring between the points chosen. This is best done by considering for a moment the case of the standard orifice and the similar loss occurring there. This we saw (Art. 59) was

$$\text{Lost head} = 0.041\ \frac{v^2}{2g},$$

the loss being expressed in terms of the velocity head found in the issuing jet.

Returning to the short tube, it seems reasonable to believe that the loss at the edge *A-B* will be closely $0.041 \times$ the velocity head in the *contracted section*, or

$$\text{Lost Head} = 0.041 \times 1.64\ h = 0.07\ h. \quad . \quad (50)$$

Substituting in the general equation the values now found for $\frac{v_m^2}{2g}$ and the lost head, and remembering that $\frac{p_a}{w} = 34$ ft., we obtain

$$0 + 34 + h = 1.64\ h + \frac{p_m}{w} + 0.07\ h,$$

or $$\frac{p_m}{w} = 34 - 0.71\ h. \quad . \quad . \quad . \quad . \quad . \quad (51)$$

That is to say, the pressure-head at the contracted section is less than atmospheric by an amount equal to $0.71\ h$. This latter is the height of the water in the tube, and we have now the relation

$$h_1 = \tfrac{3}{4}\ h \text{ (approx.)}.$$

The correctness of the relation and our assumptions was demonstrated by Venturi. With a head of 0.88 meters he obtained a water column of 0.65 meters, giving a relation of $h_1 = 0.74\ h$.

Referring to Fig. 62, it will be seen that if the distance *B-C* is less than h_1, the water will enter from the glass tube into the

mouthpiece and be expelled with the jet. This is the principle of the jet pump.

Furthermore, as the height of the water column cannot exceed 34 ft. (approx.), it follows that the pressure in the contracted section will be absolute zero when the static head is about 45 ft. (since $34 = \frac{3}{4}\ h$). Experiments conducted in the Hydraulic Laboratory of the Massachusetts Institute of Technology in 1900 showed that when the head approached or exceeded about 42 ft. the flow became "troubled and pulsatory." This was probably due to the breaking down of the contracted section when all pressure was removed.

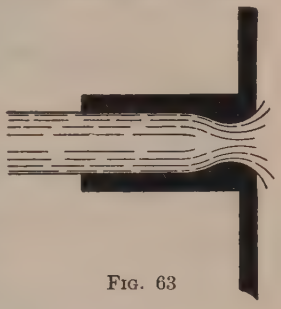

FIG. 63

At Cornell University some experiments were made on tubes which had been made of such a form internally as to conform to the shape of the contraction (Fig. 63). It was found that the flow was increased about 10 per cent. in these tubes, which may be explained by saying that the eddying would be diminished and some loss by friction prevented.

The short tube is of practical importance only as it serves later to aid in the study of flow through long pipes. As a device for measuring water, it is much less reliable than the orifice, since little is known about its coefficient, as the diameter of the tube and the head vary.

A value for the head lost in the entire tube may be obtained with the aid of equation (48) in the previous chapter. Using the value 0.82 for the tube's coefficient of velocity, we obtain

$$\text{Lost head} = \left(\frac{1}{0.82^2} - 1\right)\frac{v^2}{2g}$$

or

$$\text{Lost head} = 0.5\,\frac{v^2}{2g} \text{ (approximately).}$$

62. Conical Converging Tube.—This mouthpiece is of special interest, as it approximates the nozzle used for fire and engineering purposes. It consists of a frustum of a hollow cone with the

larger end placed at the orifice (Fig. 64). From such a tube the water flows in a jet having a slightly contracted section a-a just beyond the tip. The amount of contraction being very slight, the contraction coefficient is not far from unity and will increase as the angle of convergence β decreases. The velocity coefficient, on the other hand, is found to diminish as β decreases, and has its maximum value of 0.98 when $\beta = 180°$. The tube under this condition becomes an orifice. These variations are well shown in the accompanying table, based on the work of D'Aubisson and Castel, who experimented with this type of mouthpiece.

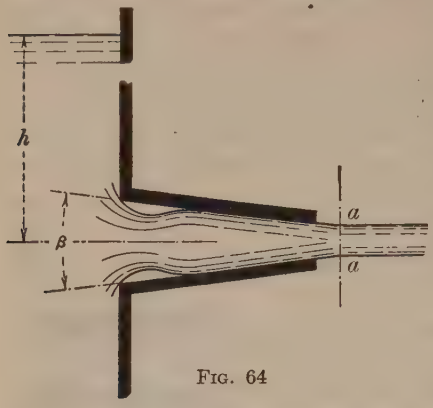

FIG. 64

ANGLE β	0° 0′	3° 10′	7° 52′	10° 20′	13° 24′	19° 28′	29° 58′	48° 50′
c_v	0.829	0.894	0.931	0.950	0.962	0.970	0.975	0.984
c_c	1.000	1.001	0.998	0.987	0.983	0.953	0.919	0.861
c	0.829	0.895	0.929	0.938	0.946	0.924	0.896	0.847

The coefficients of discharge were obtained by direct measurement of the volumes of water discharged, and the velocity coefficients were calculated by measuring the path of the jet. (See Art. 45.) The contraction coefficients were then deduced from $c_c = c \div c_v$. The method of obtaining the coefficient of velocity makes no allowance for the retardation of the particles in the jet by the atmosphere; hence the coefficients may be a little small, and the contraction coefficients would then be too large as deduced from the above equation. This would seem to account for the value of 1.001, as seen in the second column. The table would indicate that the greatest discharge occurred when the angle of convergence was near 13°, and, as the corresponding velocity was relatively high, this angle should give a very efficient nozzle. Bernoulli's Equation between a point in the

reservoir surface and a point in the contracted section of the jet shows that

$$v = c_v \sqrt{2gh}$$

and

$$Q = ca \sqrt{2gh}$$

as in the case of all orifices and tubes.

63. Nozzles for Practical Purposes.—A nozzle, as used for fire or engineering purposes, is a modification of the convergent tube, shaped to do away with the contraction in the free jet. This is generally accomplished by the addition of a cylindrical tip, as shown in Fig. 65*a*. This does not materially alter the velocity of the jet, but does increase the discharge. The tip is given a length sufficient to prevent the water passing through without wetting it. Sometimes the whole nozzle is made convex on the inside, so as to give to the particles a parallel motion as they leave the tip (Fig. 65*b*). These two types of nozzles are most commonly used in practice, as it is found that they furnish the most efficient streams. The experiments of Freeman indicate that for either kind the mean value of the discharge coefficient is very near to 0.97. For fire streams they should be particularly adapted, as they would throw large quantities of water through a maximum of distance. Among other types which have been proposed and used to a limited extent is the so-called "ring nozzle." This is made with an annular ring inside the tip Fig. 66) which causes the stream to contract as it leaves the nozzle. No good reason can be offered for its adoption, as the coefficient of discharge is made low (about 0.74) by the presence of the contraction.

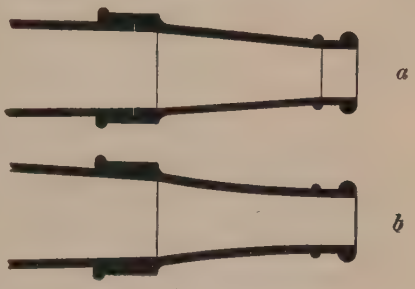

FIG. 65

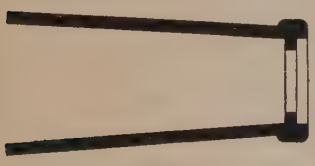

FIG. 66

For all classes of nozzles we may write for the actual velocity of the jet,

$$v = c_v \sqrt{2gh}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

and for the quantity discharged per second,

$$Q = ca\sqrt{2gh}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

where h represents the *total* head on the nozzle. When, as is usually the case, nozzles are attached to the ends of pipe lines, the total head consists of the velocity head and pressure-head in the pipe at the base of the nozzle. Writing Bernoulli's Equation between this point and the contracted section of the jet,

$$\frac{v_o^2}{2g} + \frac{p}{w} = \frac{v^2}{2g} + \left(\frac{1}{c_v^2} - 1\right)\frac{v^2}{2g}. \quad . \quad . \quad . \quad (54)$$

the last term representing the head lost in the nozzle (Art. 59), then

$$v = c_v\sqrt{2g\left(\frac{p}{w} + \frac{v_o^2}{2g}\right)}. \quad . \quad . \quad . \quad . \quad . \quad (55)$$

as was stated above. The equation gives the true value of v since the lost head was inserted in (54). Ordinarily the value of the pressure in the pipe is known, but not the value of v_o. This may be eliminated from (55) by the relation

$$Av_o = (c_c a)\, v. \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

A and a representing the area of the pipe and nozzle opening respectively. Substituting v_o from this equation and simplifying,

$$v = c_v\sqrt{\frac{2g\left(\frac{p}{w}\right)}{1 - \left(\frac{ac}{A}\right)^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

Since $Q = c_c av$,

$$Q = ca\sqrt{\frac{2g\left(\frac{p}{w}\right)}{1 - \left(\frac{ac}{A}\right)^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

The ratio $\frac{a}{A}$ may be replaced by $\left(\frac{d}{D}\right)^2$, d and D being the respective diameters.

$$v = c_v\sqrt{\frac{2g\left(\frac{p}{w}\right)}{1 - c^2\left(\frac{d}{D}\right)^4}} \quad . \quad . \quad . \quad . \quad . \quad (59)$$

$$Q = ca \sqrt{\frac{2\,g\left(\frac{p}{w}\right)}{1 - c^2\left(\frac{d}{D}\right)^4}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

These latter equations allow computations for velocity and rate of discharge if the pressure at the base of the nozzle is known. They should not be memorized as it is quite as easy to use equations (54) and (56) when solving problems. Equation (60) also readily permits the obtaining of c when Q has been obtained by an experimental run.

For computing head lost in passing a nozzle,

$$\text{Lost Head} = \left(\frac{1}{{c_v}^2} - 1\right)\frac{v^2}{2\,g}$$

as derived in Art. 59.

Example.—Compute the velocity and rate of discharge from a 2-inch nozzle, attached to a 4-inch pipe, when the pressure at the base of the nozzle during flow is 10 lb. per square inch. Assume $c_v = 0.97$ and $c_c = 0.95$.

Using equation (54),

$$\frac{{v_o}^2}{64.4} + 23 = \frac{v^2}{64.4} + \left(\frac{1}{(0.97)^2} - 1\right)\frac{v^2}{2\,g}.$$

From equation (56),

$$.087 v_o = (0.95 \times .022) v$$

or

$$v_o = 0.24 v.$$

By substitution,

$$v = 38.4 \text{ ft. per second.}$$

If the velocity head in the pipe had been neglected and $\frac{p}{w}$ been used as the head, we should have obtained

$$v = 37.4 \text{ ft. per second}$$

entailing an error of 2.5 per cent.

Note. The larger the ratio of $\frac{D}{d}$, the smaller becomes the influence of $\frac{{v_o}^2}{2\,g}$, becoming negligible when $D = 4\,d$. In this case, equations (52) and (53) may be used, h having the value of $\frac{p}{w}$.

64. Nozzle Coefficients.—A majority of nozzles having their tips shaped as in Fig. 65, contraction of the jet is done away with and $c_c = 1.00$. Obviously $c = c_v$ and for such nozzles c will have a value ranging from 0.95 to about 0.98. Freeman's experiments on the Hydraulics of Fire Streams in 1888 gave the following values for smooth conical nozzles having $c_c = 1.00$.

Diameter in Inches..	$\frac{3}{4}$	$\frac{7}{8}$	1	$1\frac{1}{8}$	$1\frac{1}{4}$	$1\frac{3}{8}$
Coefficient of Discharge.....	0.983	0.982	0.972	0.976	0.971	0.959

These coefficients were obtained when the nozzles were screwed on to the small end of a "play-pipe," 25 inches long and tapering from 2.5 to 1.55 inches, inside diameter. The large end of the play-pipe was attached to a fire hose of the same diameter, and the total head $(\frac{p}{w} + \frac{v_o^2}{2g})$ was measured at the point where play-pipe joined the hose. It would appear that the coefficients were really for play-pipe and nozzle *combined* and were smaller than what would have been determined for the nozzles alone had the head been measured at their bases. This is apparently proved by his later experiments in 1890, where using smooth nozzles on a $3\frac{3}{4}$ inch pipe and measuring the head close to the nozzles, he obtained the following values of c.

Diameter in Inches.....	1.75	2.00	2.50
Coefficient of Discharge.....	.999	.996	.997

As a device for measuring the flow of water, the nozzle ranks with the standard orifice and has the advantage of a greater capacity. Where its coefficient has been carefully determined, it can be relied upon to give discharges with a probable error of about one per cent.

65. Nozzles as Used in Engineering.—Energy of Jet.—Besides their use in connection with fire streams, nozzles are extensively used in power development and for washing away soils in mining and similar operations. For the latter purpose, smooth conical nozzles are used and jets several inches in diameter under great

heads are directed against the soil banks which are to be eroded and washed away. When used for power developments in connection with wheels or turbines, the nozzles are shaped a little differently and fitted with a movable "needle" which serves to control the nozzle opening and hence its discharge. In all the above cases it is the *Energy* in the jet that is of importance and this may be computed from the principles of Art. 39.

The velocity-head in the jet, $\frac{v^2}{2g}$, being synonymous with energy *per pound* of flowing water, the energy per second computed for any point in the jet will be the product of *energy per pound* and the *number of pounds* passing the point per second. Thus,

$$E = W \frac{v^2}{2g} \text{ ft. lb. per second.}$$

For the nozzle cited in the Example of Art. 63,

$$E = \frac{62.5 \times (0.95 \times .022)\ 38.4^3}{64.4} = 1150 \text{ ft. lb. per sec.}$$

or

$$E = \frac{1150}{550} = 2.1 \text{ H. P.}$$

66. Diverging Tubes.—It has been pointed out (Art. 61) that a part of the head lost in a short tube is occasioned by the *sudden enlargement* of the contracted stream to fill the full section of the tube. The loss may be ascribed to the sudden checking of the velocity which produces impact and violent eddying. It is a loss commonly met in hydraulic flow. If the interior of the short tube were shaped as in Fig. 67 to fit the contracted section and cause a *gradual* enlargement of the stream, this loss would be somewhat reduced and the exit velocity, and hence c_v, increased.

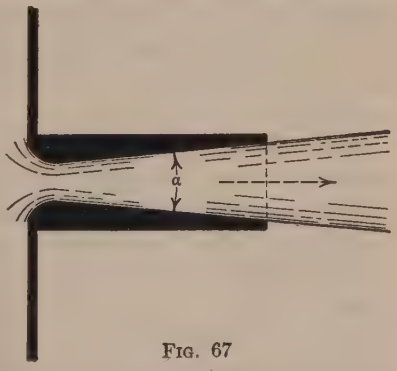

FIG. 67

Considerable experimental work has been done with tubes

shaped as in Fig. 68, the inner edge being carefully rounded to reduce eddying. Such tubes have certain noteworthy characteristics.—

(a) The value of c_v and c for a tube will be alike if the stream fills the outer end.

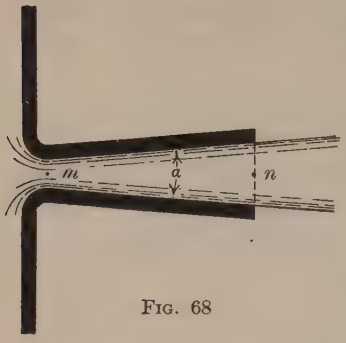

Fig. 68

(b) The pressure at the throat, or contracted section, will be less than at exit as Bernoulli's Equation between m and n will show. The velocity head at m is greater than at n, hence the pressure-head at m must be less than at n. With discharge into free air, the pressure at the throat is less than atmospheric. This will cause a velocity at this point greater than due to the head of water over the tube.

(c) If the angle of flare be made to increase, and the tube *flows full*, the difference in the velocities at m and n increases also and, as a result, the difference in the pressures at these points increases. In other words, with constant pressure at discharge, the pressure at the throat will decrease as α increases. Apparently the theoretical limit to α is reached when p_m is absolute zero. In practice it is found difficult to keep the tube flowing full when large angles are used, the stream breaking away from the sides of the tube before reaching the outer end.

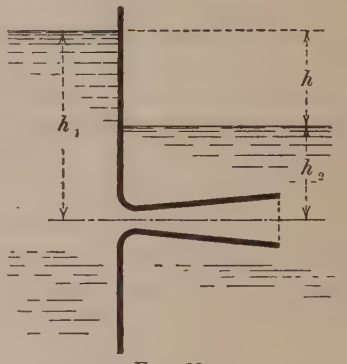

Fig. 69

(d) If the head on the tube be increased, both v_m and v_n increase in proportion, but the velocity *head* at m will increase more rapidly than that at n since the velocity head varies with the *square* of the velocity. It follows that p_m will decrease as h increases, reaching its limit at absolute zero. Experiment shows that an increase in head beyond this point results in unsteady, pulsating flow. By submerging the tube, as in Fig. 69, it has been thought possible to keep it flowing full. The effective head h in this case is $h_1 - h_2$

as it was for the submerged orifice. Experiments by the author have indicated this method to be ineffective, at least for large flares, as the stream lines confine themselves to the central portion of the tube leaving the rest filled with violent eddyings.

As a measuring device the tube is of little value as its coefficient will be found to vary widely with changes in length and angle of flare. As in previous mouthpieces we may write

$$v = c\sqrt{2gh}$$

and

$$Q = cA\sqrt{2gh},$$

the value of c being alike in each equation as before noted.

For head lost,—

$$\text{Lost Head} = \left(\frac{1}{c^2} - 1\right)\frac{v^2}{2g} \qquad \text{(See Art. 59.)}$$

The tube finds practical application in several cases. The Venturi meter for pipe lines (Art. 116), the draft tube of hydraulic turbines and the enlargements used in connecting pipes of different diameters, all make use of its principle. Comparatively little has been done to determine the range of its coefficients. Francis in his "Lowell Hydraulic Experiments" describes his work on a tube having a flare of about $4\frac{1}{3}$ degrees and various lengths. He obtained the following coefficients:

Length of Flare	12 in.	24 in.	36 in.	48 in.
Value of c.....	.70 to .78	.37 to .41	.21 to .24	.13 to .15

In 1922, urged by the importance of securing data on the friction loss in draft tubes of turbines, the author conducted a series of experiments on small tubes placed in a vertical position. The angle of flare varied from 3 to 30 degrees and various lengths were used with each flare. The diameter at the throat was constant at 1.22 inches. The head on each tube was varied from about 0.05 ft. to 1.60 ft., the tube being submerged. The discharge was weighed. The results are shown in the following table:

COEFFICIENTS FOR DIVERGING TUBES

By Russell

DIAMETER AT THROAT 1.22 INCHES

LENGTH	3°	5°	7.5°	10°	12.5°	15°	20°	25°
6 in.	0.86	0.83	0.71	0.57	0.45	0.32	0.19	0.14
12 in.	0.73	0.61	0.44	0.32	0.22	0.15	....	....
18 in.	0.62	0.45	0.29	0.18	0.12	0.08	....	....

(*a*) It will be noted that no report appears on the 30° tube. It was found impossible to expand the stream to fill this tube, the stream lines leaving its sides immediately upon passing the throat.

(*b*) The coefficients as given were fairly constant at all heads if the head exceeded 0.2 ft. Evidently this was a critical head as below it the coefficients decreased slightly with the head. Since this critical head was practically the same in value for all tubes, and the only constant dimension on the tubes was its throat diameter of 1.22 inches, it would appear that the critical head was about twice the throat diameter.

(*c*) Within the limits of heads and lengths investigated, the discharge increased with the length.

(*d*) With any given length of tube, the maximum discharge was obtained with flares of from $7\frac{1}{2}$ to 10 degrees.

Although unable to get experimental proof of the fact, the author believes it probable that none of the tubes having flares greater than 10° were filled throughout their lengths by the stream lines.

While the above values of c were obtained for small-sized tubes, it is probable that they apply to *similarly proportioned* tubes of larger size, especially if the velocity head at the throat or exit of the larger tube has limits proportional to those of the small tube.

The conical part of each tube was a separate unit attached to the rounded mouth piece by screws. Tests on the mouthpiece showed its coefficient to be 0.936 indicating a lost head of $0.14 \frac{v^2}{2g}$, a larger loss than usually assumed. The mouthpiece was made from bronze, very smooth and polished. It would appear that $0.15 \frac{v^2}{2g}$ would fairly express the loss in a rounded entrance of ordinary construction, v representing the velocity at the end of the

rounded portion. Hitherto it has been general practice to use $.05\frac{v^2}{2g}$ which appears to be too low a value.

To determine the pressure at the throat of a tube we may proceed as follows:

$$v_n = c\sqrt{2gh}$$

$$av_m = Ac\sqrt{2gh}$$

$$v_m = \frac{A}{a}c\sqrt{2gh}$$

Between reservoir level and throat,

$$h = \frac{{v_m}^2}{2g} + \frac{p_m}{w} + \frac{.15{v_m}^2}{2g}$$

From the last two equations v_m may be eliminated, giving the value of p_m when c is known, or c when p_m is known.

67. Borda's Mouthpiece.—Mention is made here of this mouthpiece, not because of its practical importance as a measuring device, but because it presents the only case allowing a theoretical determination of the contraction coefficient. As shown in Fig. 70, it consists of an orifice fitted with a short re-entrant tube having a length about equal to its diameter. The issuing jet touches only the inner edge. To determine the amount of the contraction we may proceed as follows:

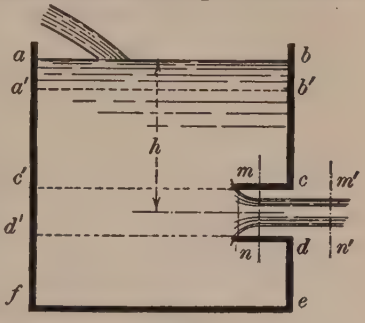

FIG. 70

On the portions *bc* and *de* of the reservoir walls, the pressure at any point is practically the same as it would be under static conditions, since the presence of the tube prevents the particles in close proximity to the walls from having any appreciable velocity. The total pressure on *af* is therefore in excess of that on *be* by the amount, *awh*, that would exist on *c'd'*, the projected area of the orifice. Considering now the mass of water between *ab* and *mn*, we have this same unbalanced pressure exerted upon it by the wall *af*. If in a time *dt* the mass moves to *a'b'm'n'* and in so doing has its momentum in a horizontal direction changed, we may

equate this change to the product of the horizontal force, awh, which produced it, and the time in which it was produced. That is,

$$awhdt = Mdv.$$

The change in momentum may be expressed as the difference in horizontal momentum of the small masses $aba'b'$ and $mnm'n'$. That this is so may be seen if it is noted that the aggregate momentum of all particles between $a'b'$ and mn remains constant. (A more detailed proof might be given similar to the discussion in Art. 36 relative to the change in kinetic energy taking place in the mass A-B, Fig. 39.) For $aba'b'$ the horizontal momentum is zero, and that of $mnm'n'$ is

$$\text{Mass} \times \text{velocity} = \frac{w}{g}(c_c a)\, vdt \times v.$$

We may therefore write

$$(awh)\, dt = \frac{w}{g} c_c a v^2 dt,$$

from which $$c_c = h \div \frac{v^2}{g} = \frac{v^2}{2g} \div \frac{v^2}{g} = 0.5.$$

This value is a theoretical one, since in substituting $\frac{v^2}{2g}$ for h no allowance was made for loss in velocity due to friction. If c_v be assumed as 0.98, we obtain

$$c_c = 0.52,$$

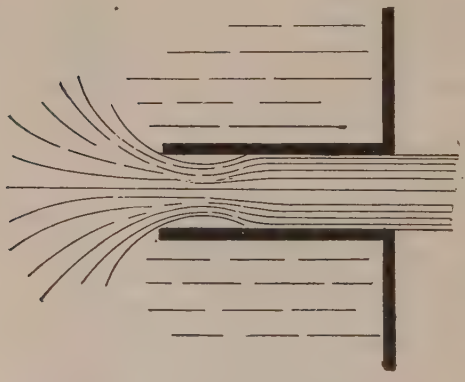

Fig. 71

showing that the amount of contraction is greater than in the standard orifice (Art. 43). This is doubtless due to the change, brought about by the sides of the tube, in direction of the water particles as they approach the opening.

68. Re-entrant Tubes.—If the tube of Fig. 70 be lengthened to about 3 diameters (Fig. 71) the stream may be made to expand again, filling the tube at exit as in the case of the

standard short tube. The contraction at the inner end will be greater than in the standard short tube and since the subsequent expansion is greater, the lost head in the tube will be increased and c_v for the tube will be diminished. Experiment indicates a value of $c_v = 0.75$ for tubes larger than $2\frac{1}{2}$ inches diameter. Using this value,

$$\text{Lost Head} = \left[\frac{1}{(.75)^2} - 1\right]\frac{v^2}{2g} = 0.78\frac{v^2}{2g}. \quad . \quad . \quad (61)$$

A condition of flow, quite similar to that through the re-entrant tube, occurs when a pipe is inserted through a reservoir wall and made to project inwardly. In this case the loss in head at the entrance of the pipe is assumed to be equal to that expressed in (61).

PROBLEMS

1. Water is discharged from a reservoir through a standard short tube under a head of 30 ft.

(*a*) What is the velocity head in the stream at exit?

(*b*) What is the velocity head in the contracted section?

(*c*) What is the absolute pressure-head at the contracted section?

Ans. (*a*) 20.2 ft.
(*b*) 49.5 ft.
(*c*) 12.7 ft.

2. A short cylindrical tube is attached to the vertical wall of a reservoir, but differs from a standard short tube in that it projects into the reservoir a distance of half its length. This causes the amount of the contraction at entrance to be increased and experimental observations indicate a coefficient of contraction at this point of 0.50. The coefficient of velocity for the whole tube being 0.72, determine what value the head on the tube must be in order that the pressure in the contracted vein shall be absolute zero.

3. Compute the values of the total loss in head for Problems 1 and 2.

4. A 2-inch nozzle is attached to a 6-inch pipe. Pressure at the base of the nozzle during the flow is 50 lb. per square inch. If the coefficient of velocity and discharge are both 0.90, find

(*a*) Velocity and rate of discharge.

(*b*) Energy per second in water discharged.

Ans. (*a*) 77.7 ft. per second.
1.70 cu. ft. per second.
(*b*) 9,900 ft-lb. per second.

5. A $1\frac{1}{4}$-inch nozzle, attached to a $2\frac{1}{2}$-inch pipe, discharges 290 gal. per minute under a pressure of 40 lb. per square inch at the base of the nozzle. What is the coefficient of discharge of the nozzle?

Ans. 0.96.

6. A nozzle points vertically downward and terminates in a $1\frac{3}{8}$-inch orifice. It is supplied by a $2\frac{1}{2}$-inch pipe, to which is attached a pressure gauge 3 ft. above the nozzle orifice. When the gauge registers 30 lb. per square inch the discharge is found to be 310 gal. per minute. What head is being lost between the gauge and the orifice? Assume coefficient of contraction at exit to be 1.00.

Ans. 8.3 ft.

7. A $1\frac{1}{4}$-inch nozzle, attached to a horizontal $2\frac{1}{2}$-inch pipe, has a coefficient of discharge of 0.95. If, during flow, the gauge pressure in the pipe be 80 lb. per square inch, what should be the discharge in gallons per minute?

Ans. 398 gal. per minute.

8. The nozzle which furnishes the water to a certain hydraulic turbine is 11 inches in diameter and has a coefficient of velocity and discharge of 0.975 and 0.960 respectively. The nozzle is supplied from a 24-inch pipe in which the water approaches the nozzle with a *total* head of 1100 ft. Compute the velocity and rate of discharge from the nozzle, also the energy per second delivered by the jet to the turbine. What energy, measured in H. P., is lost in passing through the nozzle? What is the efficiency of the nozzle?

9. A horizontal diverging tube, such as shown in Fig. 70, has a diameter of 3 in. at the contracted section and 4 in. at its mouth. Water stands 16 ft. deep above its axis on the reservoir side and 2 ft. deep on its discharging end. Determine the pressure existing in the contracted section assuming that the only loss in the tube is one of 2 ft. occurring between the contracted section and the point of discharge.

Ans. 5.3 lb. per sq. in.

10. If in the previous problem a vertical tube, 1 inch in diameter, were tapped into the contracted section and made to communicate with a second reservoir whose level was 8 ft. below the axis of the diverging tube, at what velocity would water flow from the vertical tube into the large one, assuming no change in pressure at the point of entrance?

Ans. 29.9 ft. per second.

11. A horizontal diverging tube connects two reservoirs. Its inlet end is rounded (entrance loss negligible) and is 10 ft. below the surface of the

supplying reservoir. The diameter at the small section is 2 inches and the discharge end is 3 ft. below the water level of the receiving reservoir. What value for the diameter of the discharging end will be consistent with a maximum rate of flow through the tube? Assume a loss of 1 ft. in passing from the contracted section to the exit.

Ans. 3 .30 in.

12. Determine the pressure at the throat of one of Russell's tubes, which had a length of 12 inches and a flare of 10 degrees, when discharging under a head of 1 ft. Compute the head lost in passing through the tube.

13. A horizontal diverging tube discharges from a reservoir under a head of 6 ft. If the diameter at the large end be 5 inches and the discharge coefficient 0 .73, what diameter at the throat will be necessary to produce an absolute pressure there of 5 lb. per sq. inch? Assume the loss at entrance to be 0 .15 of the velocity head at the contracted section.

14. A diverging tube, 6 ft. long, axis vertical, is attached to the bottom of a reservoir. During flow, the pressure at the throat, which is 2 ft. in diameter, is $\frac{3}{4}$ of an atmosphere, and the velocity is 30 ft. per second. The losses in the tube may be assumed as follows:

At entrance $= \frac{1}{10}$ the velocity head in throat.

Between throat and exit $= \frac{1}{3}$ the velocity head at exit. Compute the diameter at exit and the height from point of exit to reservoir surface.

15. A short re-entrant tube (similar to Borda's mouthpiece), 3 in. in diameter, is fitted in the vertical side of a reservoir whose superficial area is 22 sq. ft. What time will be required to lower the water level 2 ft. from an initial head of 8 ft.?

Ans. 83 .3 sec.

LITERATURE

John R. Freeman: Experiments Relating to the Hydraulics of Fire Streams. Trans. Amer. Soc. Civ. Engrs., vol. 21, 1889.

J. T. Fanning: Experiments on Streams from Fire Nozzles. Engineering News, July 14, 1892.

CHAPTER VII

FLOW OVER WEIRS

69. Definitions.—As generally understood, the term *Weir* is used to designate a notched opening made in the upper edge of a vertical wall, through which water is allowed to flow for purpose of measurement. Overflow dams and spillway sections in dry-crested dams are sometimes spoken of as weirs since they offer means for an approximate determination of the rate of discharge over them. They will be separately and later discussed. As usually constructed, the weir has a simple geometrical shape and the notch is sharp-edged as in the case of the standard orifice, so that the stream touches only a line. The opening is usually rectangular, triangular or trapezoidal. In each case the edge of the opening over which the water flows is called the *Crest*, and its height above the bottom of the reservoir or channel is known as the *Crest Height*. The French term *Nappe* (Sheet) is often applied to the overfalling stream of water.

70. The Rectangular Weir.—This is the more common type in use, the sides of the notch being horizontal and vertical. The horizontal side alone is the *crest*. If the bottom and sides of the notch be far enough removed from the bottom and sides of the reservoir to permit free lateral approach of the water in the plane of the weir, the stream issues from the notch contracted on these three sides and we have what is known as the *Contracted Weir* (Fig. 72). If the length of the crest be extended, as in Fig. 73, so as to make the sides of the notch coincident with the side walls of the reservoir or channel, the side or *end* contractions will be suppressed and the weir is known as a *Suppressed* Weir. In both weirs, the surface of the water over the crest, and immediately back of it, assumes a curve as indicated in Fig. 74,

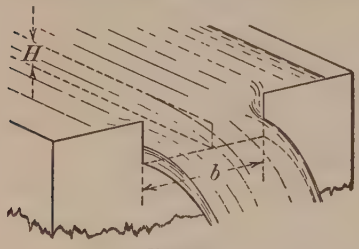

FIG. 72

forming the *Surface contraction.* The *Head,* H, is the vertical distance from the level of the crest up to the general reservoir surface at a point where the latter is unaffected by the surface contraction.

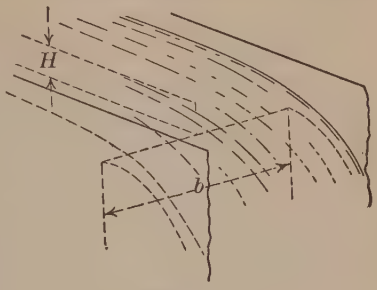

Fig. 73

If a large, rectangular orifice discharges with a head on its upper edge equal to zero, we have results analogous to the flow over the rectangular weir. Let Fig. 75 represent in section such an orifice placed in the end wall of a reservoir or channel, the latter so shaped or proportioned as to produce a velocity of approach v_o. If v represents the velocity of the discharge through any imaginary, horizontal, elementary strip situated a distance x below the surface, the discharge per second is

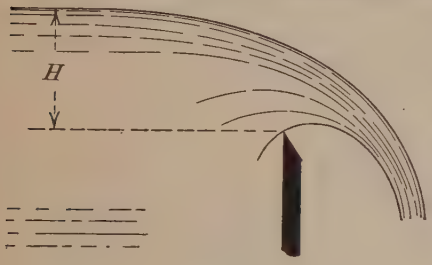

Fig. 74

$$dQ = dA \,.\, v = b \,.\, dx \,.\, v,$$

and for the entire orifice,

$$Q = b \int v \,.\, dx$$

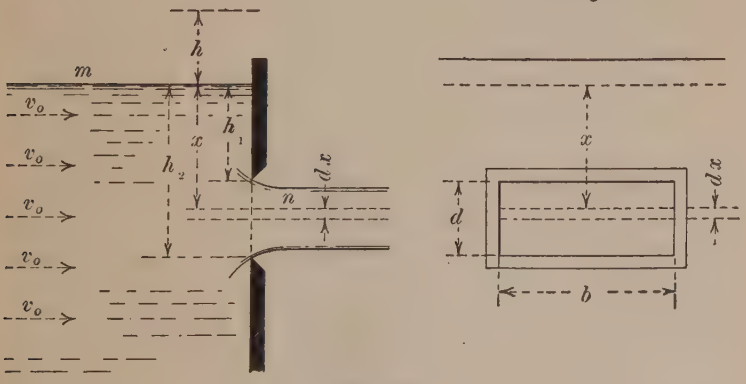

Fig. 75

To express v in terms of x, Bernoulli's Theorem may be written between the points m and n, giving

$$\frac{v_o^2}{2g} + \frac{p_a}{w} + x = \frac{v^2}{2g} + \frac{p_a}{w} + 0,$$

and

$$v = \sqrt{2g}\,(x + h)^{\frac{1}{2}},$$

where h has replaced $\frac{v_o^2}{2g}$. Substituting this value for v in the above expression for Q, there results

$$Q = b\sqrt{2g}\int (x + h)^{\frac{1}{2}} dx.$$

The quantity $(x+h)$ may be regarded as a variable with limits of (h_1+h) and (h_2+h) inasmuch as $d(x+h) = dx$. With these limits the above equation becomes, when integrated,

$$Q = \tfrac{2}{3}\, b\sqrt{2g}\,[\,(h_2 + h)^{\frac{3}{2}} - (h_1 + h)^{\frac{3}{2}}].$$

By making h_1 equal to zero, the orifice becomes a weir, and, replacing h_2 by H, we obtain

$$Q = \tfrac{2}{3}\, b\sqrt{2g}\,[\,(H + h)^{\frac{3}{2}} - h^{\frac{3}{2}}]. \quad . \quad . \quad . \quad . \quad (62)$$

If there be no velocity of approach, $h=0$ and

$$Q = \tfrac{2}{3}\, b\sqrt{2g}\, H^{\frac{3}{2}}. \quad . \quad . \quad . \quad . \quad . \quad (63)$$

In deriving (62) and (63), no allowance for friction or contraction of the stream was made and these two equations may, therefore, be regarded as *fundamental* equations for the discharge from either the Contracted or Suppressed weir. It should be noted that the value of v_o has been assumed uniform in all parts of the channel section and hence has the value, $v_o = \frac{Q}{A}$, where A is the area of the cross-section. The fact that v_o is *not* uniform leads, under some conditions, to a discrepancy between Q as actually observed and as computed by formulas based on (62) or (63). If a coefficient, c, be inserted in the fundamental equations to correct for frictional resistances and the diminution in discharge due to contraction, the equations became practical formulas. To ascertain the value of c under different conditions of head, length and type of weir has been the aim of many experimenters. They have tried also to deduce other formulas with an aim to simplicity and wide range of application. A discussion of the best of these attempts will be given, also probable reasons for the occasional disagreement among results obtained from the proposed formulas.

Before considering the formulas in detail it will be helpful to discuss some of the general conditions present in weir flow and their effect on discharge.

(a) *Contractions of the nappe.* Just as in the case of the orifice, the presence of the contractions diminishes the discharge. If the amount of the contraction were constant under all *heads, lengths of crest,* and for all *velocities of approach,* a few careful experiments would serve to fix its numerical value, and that of the coefficient of discharge, for use with our fundamental equations (62) and (63). The opposite appears to be the case and no exact relation between these variables and c has ever been obtained. The difficulty has usually been met either by assigning to c a fixed *average* value or making it a function of the head, H.

(b) *Velocity of approach.* This quantity makes itself felt in at least three different ways.

First, it increases the discharge as our previous reasoning has shown; but in assuming, as we did, that the kinetic energy of all particles as they approach the weir is represented by $\frac{v_o^2}{2g}$, we are quite wrong if we give to the latter the mean value of $\frac{Q}{A}$. We know that the water nearer the surface and center of the channel has a velocity greater than the mean, and certainly it is the kinetic energy of this water that is going to affect the discharge more than that of more remote and slowly moving particles which must have their directional velocity changed in approaching the crest. It would appear that the value of h as computed from the *mean* velocity of approach is really too small and that some value αh should be used, α having a value greater than unity.

Second, the velocity of approach if large, must have some effect on the top, bottom and end contractions of the nappe, and, as previously stated, just what this is we do not know.

Third, the existence of velocity, however small, in the channel means that the surface of the water is inclined in the direction of motion, or toward the crest. The head on the weir being measured from the horizontal plane of the crest up to the water surface, it follows that slightly different values will be obtained according to whether the measurement be made a few feet from the crest or at a greater distance. If the velocity be small, then the slope may

be so small that it will not affect the measurement of H; but with high velocities it probably has some effect.

The student would do well to bear the above facts in mind as he studies and compares the proposed formulas of the various investigators.

71. Francis' Formulas.—During the years between 1848 and 1852, James B. Francis carried on at Lowell, Mass., an extensive series of experiments with rectangular weirs. The length of crest varied from 3.5 feet to 17 feet, but in the majority of the experiments, the length was practically 10 feet. The head, H, varied from 0.6 to 1.6 feet and the volume discharged was measured in a large collecting basin immediately below the weir. As a result of over eighty experiments, he proposed the two following equations:

$$Q = 0.622 \times \frac{2}{3}\left(b - \frac{nH}{10}\right)\sqrt{2g}\left[(H + h)^{\frac{3}{2}} - h^{\frac{3}{2}}\right], \qquad (64)$$

and

$$Q = 0.622 \times \frac{2}{3}\left(b - \frac{nH}{10}\right)\sqrt{2g}\, H^{\frac{3}{2}}, \quad . \quad . \quad . \quad (65)$$

as being adapted to rectangular weirs, with or without end contractions. The first (64) provides for the effect of the velocity of approach, and the second is to be used when this is so small as to be negligible. It will be noticed that they are very similar in form to our fundamental equations (62) and (63), the effect of velocity of approach being cared for in the same way. In studying the effect of contraction, Francis was led to conclude that the coefficient of *vertical* contraction was constant at all heads, while the coefficient of *end* contraction (when end contraction was present) varied with the head. He accordingly divided his coefficient into two parts, the first having a value of 0.622, an average value which in over eighty experiments differed from the extreme values by about 3 per cent. To provide for the effect of the end contractions, he concluded that since the amount of contraction varied with the head, H, the diminution in discharge caused by it was constant for any head and had the effect of decreasing the effective length of the weir. He, therefore, proposed that $\left(b - \frac{nH}{10}\right)$ represent the effective length of the weir, the quantity $\frac{H}{10}$ being wholly empirical

and derived from his data, and the value of n representing the number of end contractions present. Since $n=2$ for the Contracted Weir and zero for the Suppressed Weir, we may write our equations as follows:

(*a*) *Contracted Weir.*—

$$Q = 3.33\left(b - \frac{2H}{10}\right)\left[(H+h)^{\frac{3}{2}} - h^{\frac{3}{2}}\right], \text{ with vel. of approach.} \quad (66)$$

$$Q = 3.33\left(b - \frac{2H}{10}\right)H^{\frac{3}{2}}, \text{ no vel. of approach.} \quad (67)$$

(*b*) *Suppressed Weir.*—

$$Q = 3.33\,b\left[(H+h)^{\frac{3}{2}} - h^{\frac{3}{2}}\right], \text{ with vel. of approach.} \quad (68)$$

$$Q = 3.33\,bH^{\frac{3}{2}}, \text{ with no vel. of approach.} \quad (69)$$

If for any reason, the contraction at one end only is suppressed, then $n=1$.

Francis also found that the curvature of the end contractions extended horizontally over a distance of about $1.5\,H$, and consequently recommended that, for a weir with two end contractions, the length b be greater than $3\,H$. Otherwise, the effect of end and vertical contractions would not be properly cared for by his coefficient 0.622 and the quantity $\left(b - \frac{nH}{10}\right)$.

A direct solution of equation (66) or (68) necessitates a knowledge of h whose value cannot be determined until Q and v_o are known. A solution by the method of trial is possible, the approximate value of Q being determined by (67) and (69). Using this value, v_o and then h may be computed and the more exact equations (66) and (68) used to give a new value of Q. This latter will still be somewhat approximate if the value of v_o be considerable, and it must be again used to determine a new v_0 and h to give a value of Q which will not differ from the previous one sufficient enough to warrant a recalculation of h. This method of correcting for the velocity of approach should be strictly adhered to in using Francis' formulas, since it was the method used by him when determining the value of his coefficient. The substitution of $(H + \alpha h)$ for the effective head should not be made.

Example.—A suppressed weir having a crest 10.58 feet long discharges under a head of 0.682 feet. If the depth of the channel of approach be 2.2 feet, find the discharge per second. Neglecting the velocity of approach,

$$Q = 3.33 \times 10.58 \times (0.682)^{\frac{3}{2}}$$

$$Q = 19.84 \text{ cu. ft. per second.}$$

$$v_o = \frac{Q}{A} = \frac{19.84}{10.58 \times 2.2} = 0.85 \text{ ft. per second.}$$

$$h = \frac{(0.85)^2}{64.4} = 0.011 \text{ ft.}$$

Using formula (68),

$$Q = 3.33 \times 10.58 \left[(0.693)^{\frac{3}{2}} - (0.011)^{\frac{3}{2}} \right]$$

$$Q = 20.3 \text{ cu. ft. per second.}$$

This may be taken as the final value, inasmuch as a recalculation of h gives 0.012, and this differs so slightly from the value used as to make resubstitution unnecessary.

72. Fteley and Stearns Formulas.—In 1877–79, Alphonse Fteley and Frederick P. Stearns experimented with rectangular weirs at Framingham, Mass., using weirs 5 feet and 19 feet long. From over fifty experiments on the 5 ft. weir *with end contractions* they concluded that the end contraction did not vary directly as the head and that the general proportions of the weir influenced it. In other words, they did not find that $\frac{H}{10}$ in Francis' formulas expressed exactly the effect of contraction. They could not, however, discover the law that governed the amount of it. As a result, they recommended the use of weirs without end contractions, believing that the latter were a source of error.

From the result of work on suppressed weirs, 5 and 19 ft. long, they proposed

$$Q = 3.31\, bH^{\frac{3}{2}} + 0.007\, b, \quad . \quad . \quad . \quad (70)$$

to be used when the velocity of approach is negligible, and

$$Q = 3.31\, b\, (H + 1.5\, h)^{\frac{3}{2}} + 0.007\, b, \quad . \quad . \quad . \quad (71)$$

when it is so large as to influence the discharge.

It will be noticed that the correction for velocity of approach is made by adding αh to the measured H according to the method and reasoning explained in Art. 70. The value of 1.5 was stated by the experimentors to be an *average* value, α varying slightly with the ratio of H to crest height. More exact values are furnished in the following table:

VALUES OF α FOR USE IN FTELEY AND STEARNS' SUPPRESSED WEIR FORMULA

HEAD ON WEIR	DEPTH OF CHANNEL OF APPROACH BELOW CREST.			
	2.60	1.70	1.00	0.50
0.20	1.51	1.66	1.87	1.70
0.30	1.50	1.65	1.83	1.53
0.40	1.49	1.63	1.79	1.53
0.50	1.48	1.62	1.75	1.53
0.60	1.47	1.60	1.71	1.52
0.70	1.46	1.59	1.68	1.51
0.80	1.45	1.57	1.65	1.50
0.90	1.44	1.56	1.63	1.49
1.00	1.43	1.54	1.61	1.48
1.10	1.42	1.53	1.59	
1.20	1.41	1.51	1.57	
1.30	1.40	1.49	1.55	
1.40	1.39	1.48	1.54	
1.50	1.38	1.46	1.52	
1.60	1.37	1.44	1.51	
1.70	1.36	1.43	1.49	
1.80	1.35	1.41		
1.90	1.34	1.40		
2.0	1.33	1.38		

Although recommending the avoidance of weirs with end contractions, from their work on this type of weir they proposed

$$Q = 3.31\,(b - \frac{2}{10}H)\,(H + 2.05\,h)^{\frac{3}{2}} + 0.007\,b. \qquad (72)$$

the value 2.05 being a mean value, and the correction for end contractions being according to Francis' method.

The range of head in the Fteley-Stearns experiments was from 0.07 to 1.63 feet.

73. Bazin's Formula for Suppressed Weirs.—In 1886 H. Bazin of France began a series of experiments on weir flow that

have since received wide recognition on account of the masterly way in which he carried them out. His experiments and studies led him to propose for a suppressed weir, *having no velocity of approach*,

$$Q = cb\sqrt{2g}\,H^{\frac{3}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

and for c he deduced the value,

$$c = 0.405 + \frac{0.00984}{H},$$

stating that for heads greater than 0.33 feet this value was sufficiently precise for practical work. The form of (73) is identical with our fundamental equation (63).

To correct for velocity of approach, Bazin adopted the method of adding ah to the measured head, as did Fteley and Stearns, and from his own work and that of these investigators he adopted a mean value for a of 1.69. The value of h he expressed in terms of the height of crest, P, as follows:

$$v_o = \frac{Q}{A} = \frac{Q}{(P+H)b}$$

$$h = \frac{v_o^2}{2g} = \frac{Q^2}{2g\,(P+H)^2b^2}$$

Using this value of h and adding 1.69 h to H in equation (73), Bazin finally obtained

$$Q = \left[0.405 + \frac{0.00984}{H}\right]\left[1 + 0.55\left(\frac{H}{P+H}\right)^2\right] b\sqrt{2g}\,H^{\frac{3}{2}}. \quad (74)$$

The first bracket quantity is the coefficient c and the second is the velocity correction. For heads between 0.33 and 1.0 feet he states that we may use

$$Q = \left[0.425 + 0.212\left(\frac{H}{P+H}\right)^2\right] b\sqrt{2g}\,H^{\frac{3}{2}}, \quad . \quad . \quad (75)$$

with an error less than 3 per cent. The interesting fact concerning (74) is that it provides for the influence of the velocity of approach

and allows Q to be calculated directly without resorting to the method of trial.

Bazin's weirs ranged from 1.64 to 6.56 feet in width, and the head from 0.164 to 1.969 feet. The value of P varied from 0.66 to 6.56 feet.

74. Formulas of Hamilton Smith.—In 1886 Hamilton Smith, a noted American hydraulic engineer, published in his "Hydraulics" a very careful résumé and compilation of the best and most trustworthy experiments on weirs. From a study of the work of Francis, Fteley and Stearns, and others, he proposed for

Contracted Weirs.—

$$Q = c \times \tfrac{2}{3}\, b \sqrt{2g}\, H^{\frac{3}{2}}, \text{ with no vel. of approach.} \quad . \quad . \quad (76)$$

$$Q = c \times \tfrac{2}{3}\, b \sqrt{2g}\, (H + 1.4\, h)^{\frac{3}{2}}, \text{ with vel. of approach.} \quad (77)$$

and for

Suppressed Weirs.—

$$Q = c \times \tfrac{2}{3}\, b \sqrt{2g}\, H^{\frac{3}{2}}, \text{ with no vel. of approach.} \quad . \quad . \quad . \quad (78)$$

$$Q = c \times \tfrac{2}{3}\, b \sqrt{2g}\, (H + \tfrac{4}{3}\, h)^{\frac{3}{2}}, \text{ with vel. of approach.} \quad . \quad (79)$$

Formulas (76) and (78) are identical with the fundamental equation (63), having values of c dependent on the type of weir and varying with its length and effective head $(H + ah)$. As for his mode of correcting the head to care for the velocity of approach, we see it is similar to that used by Fteley and Stearns. The difference between $1.4\,h$ as used in (77) and $\frac{4}{3}h$ appearing in (79) can probably be explained by saying that a weir with two end contractions is generally placed near to the center of the channel where the water is moving with its highest velocity.

The accompanying tables give values of c for a wide range of conditions. The first column in each table is the effective head $(H + ah)$, but in selecting a coefficient, no appreciable error will result if we use H instead, since ah is very small. For heads and lengths not given c must be found by interpolation.

Values of c in Smith's Contracted Weir Formula

Effective Head in Feet	Length of Weir in Feet						
	0.66	1	2	3	5	10	19
0.1	0.632	0.639	0.646	0.652	0.653	0.655	0.656
0.15	.619	.625	.634	.638	.640	.641	.642
0.2	.611	.618	.626	.630	.631	.633	.634
0.25	.605	.612	.621	.624	.626	.628	.629
0.3	.601	.608	.616	.619	.621	.624	.625
0.4	.595	.601	.609	.613	.615	.618	.620
0.5	.590	.596	.605	.608	.611	.615	.617
0.6	.587	.593	.601	.605	.608	.613	.615
0.7		.590	.598	.603	.606	.612	.614
0.8			.595	.600	.604	.611	.613
0.9			.592	.598	.603	.609	.612
1.0			.590	.595	.601	.608	.611
1.2			.585	.591	.597	.605	.610
1.4			.580	.587	.594	.602	.609
1.6				.582	.591	.600	.607

Values of c in Smith's Suppressed Weir Formula

Effective Head in Feet	Length of Weir in Feet							
	0.66	2	3	4	5	7	10	19
0.1	0.675				0.659	0.658	0.658	0.657
0.15	.662	0.652	0.649	0.647	.645	.645	.644	.643
0.2	.656	.645	.642	.641	.638	.637	.637	.635
0.25	.653	.641	.638	.636	.634	.633	.632	.630
0.3	.651	.639	.636	.633	.631	.629	.628	.626
0.4	.650	.636	.633	.630	.628	.625	.623	.621
0.5	.650	.637	.633	.630	.627	.624	.621	.619
0.6	.651	.638	.634	.630	.627	.623	.620	.618
0.7	.658	.640	.635	.631	.628	.624	.620	.618
0.8	.656	.643	.637	.633	.629	.625	.621	.618
0.9		.645	.639	.635	.631	.627	.622	.619
1.0		.648	.641	.637	.633	.628	.624	.619
1.2			.646	.641	.636	.632	.626	.620
1.4				.644	.640	.634	.629	.622
1.6				.647	.642	.637	.631	.623

75. Barnes' Formulas.—In 1916, Alfred A. Barnes of Manchester, England, published in his "Hydraulic Flow Reviewed" a most interesting discussion of the experimental data obtained by Lesbros (1827), Castel (1835) and Francis (1851), upon *Contracted Weirs*, and proposed

$$Q = 3.324\left(H + \frac{v_o^2}{70}\right)^{1.49} b^{1.11}\left[b + 2\left(H + \frac{v_o^2}{70}\right)\right]^{-0.11}. \quad (80)$$

in which H is the measured head and v_o the mean velocity of approach in the channel. Since the latter is generally unknown at first, Q is approximately obtained by omitting the term $\frac{v_o^2}{70}$. The value of v_o is then obtained from

$$v_o = \frac{Q}{A}$$

and equation (80) used to get a closer value of Q. If the latter differs much from the value used in obtaining v_o, a new v_o may be calculated and (80) used once more. The method is similar to that employed in the Francis equations.

The data studied included weirs from 8 inches to 10 feet in length and the head ranged from $\frac{3}{4}$ inch to 19 inches. Barnes found that the above formula fitted the *observed* discharges with an error less than one per cent in 80 out of 82 experiments. In only ten instances was the error greater than one-half per cent. and six of these occurred when the crest height was so low as to possibly affect contraction. A noteworthy thing is the apparent accuracy of the formula for a head as low as $\frac{3}{4}$ inch.

For discharge from a Suppressed Weir, Barnes proposed

$$Q = 3.324\left(H + \frac{v_o^2}{50}\right)^{1.49} b^{0.98}\left[b + 2\left(H + \frac{v_o^2}{50}\right)\right]^{0.02}. \quad (81)$$

This was obtained from a study of the work of Francis, Fteley and Stearns, Bazin, Boileau and Rafter, covering weir lengths from 2.5 feet to 19 feet, and heads from 0.12 feet to 6.0 feet. The heads above 1.6 feet were from data by Rafter obtained at Cornell University. When compared with 92 experiments, the greatest error arising from use of (81) was 1.7 per cent and in 73 cases the formula agreed within 1 per cent.

The value of v_o is obtained as in equation (80). Both of the above formulas represent an attempt to break away entirely from the

fundamental equations (62) and (63) and substitute equations which represent the best curves that can be constructed upon plotted data. They are largely empirical, although they have a rational basis. Apparently they apply throughout a great range in head and length of crest and, if subsequent investigations prove their general accuracy, they must be regarded as very valuable. The fact that the exponential values are unusual should deter no engineer from their use as they are readily solved by logarithms.

76. General Agreement of Formulas.—Gibson, in his "Hydraulics and its Applications," gives the following short table, showing the relative results of applying the various formulas to a suppressed weir 10 feet long, discharging with different values of head and velocity of approach.

Length	Crest Height	Head	Velocity Approach	Francis	Fteley-Stearns	Bazin	Smith
10 ft.	4 ft.	1.0 ft.	.68 f. s.	1.00	1.001	1.010	1.002
10 ft.	2 ft.	1.0 ft.	1.16 f. s.	1.00	1.015	.985	1.018
10 ft.	4 ft.	4.0 ft.	2.15 f. s.	1.00	1.050	1.050	1.115

For the purpose of comparison, the quantity as obtained by the Francis formula is taken as the "unit of comparison" and the quantities are expressed in terms of this unit.

A careful inspection of the table shows that all the formulas agree fairly well *when the velocity of approach is low in value.* As the latter increases, the agreement is less close and for the last weir considered, there is a wide discrepancy in results.

If to the table be added the results of Barnes' formula we should have 1.005, 1.01 and 1.004 for the three comparative values of the discharge. The agreement with Francis is quite marked, and it will be found that in general these two agree very well save for extremely low heads where the Barnes' formula gives slightly larger results.

77. Choice of Formula.—It is impossible to say that one certain formula is better than the others for general use. In selecting a formula for a given weir, a careful study of its proportions and of the velocity of approach should be made and then that formula selected which was derived under nearly similar conditions.

However, this is not always easy to do owing to the many possible combinations of crest length, crest height, head and velocity of approach. Where it is possible to do so, it would seem wise to closely duplicate the weir and conditions of any one investigator and then use his formula in computing the discharge.

It has been the general practice in this country to use the Fteley and Stearns formulas for heads from 0.07 to 0.6 feet and the Francis' for heads higher than this. The Bazin formula has not been so popular here as in England and France, but it should be regarded as particularly useful for short weirs and those having considerable velocity of approach. For weirs not conforming to the general proportions or conditions of any of the above investigators, the author believes that Barnes' formulas are the most satisfactory since they were based on a consideration of widely different data.

78. Measuring the Head.—All the experimenters above mentioned have measured the head by observing the height of the water level in an open pail or box which communicated with the water in the channel by means of a tube inserted through the channel wall and having its end flush with and normal to the plane of the wall's inner surface. This, therefore, becomes the standard method and should not be departed from if accuracy is being sought. Francis' piezometer was inserted at a point 6 feet upstream from the crest and about six inches lower than the crest. Fteley and Stearns also located their connection 6 feet upstream but its elevation with reference to the crest varied, being 0.4, 2.4 and 0.0 feet below the latter in the several experiments. Bazin's piezometer was inserted at a point 16.4 feet upstream and close to the bottom of the channel and it is not at all improbable that this greater distance might have resulted in larger heads being observed when the velocity of approach was large.

The height of the water level in the pail or box was determined by a hook gauge (Art. 55), the reading of the gauge, when its point was in the same horizontal plane as the crest, being previously determined. This constitutes the standard mode of measuring head and should be used in all cases where accuracy is sought unless the head is very large. For small heads there is no other reliable means. To determine the "initial reading" of the gauges, we may proceed in several ways. Where possible, the water in the channel may be by-passed or its flow checked until the level has

fallen to that of the crest. Capillary attraction may cause a curvature in the water surface close to the crest and the water surface appear to be level with the crest when in reality it is below it. The proper level can be accurately determined by observing the reflections of nearby objects in the water surface close to the crest. If the reflections are free from all distortion at this point the surface must be a plane and level with the crest.

A preferable method and one successfully used by the writer consists in making use of an engineer's level and a very light level-rod to set the point of the hook at the same elevation as the crest. Using a carefully adjusted level, with a sensitive bubble, and a rod obtained by borrowing the center slide of an ordinary 10-inch calculating rule, it was found possible, by repetition, to set the point of the hook at the elevation of the crest with an error entirely negligible. The light rod was alternately placed on the crest and the point of the hook, the latter being lowered or raised to give the same rod reading as the crest. The reading of the vernier on the gauge was noted for each setting and the mean calculated.

For coarser grades of work and for large heads, a vertical staff gauge, graduated to hundredths of a foot, may be set in the gauge box or even in the stream itself, provided the surface be kept quiet. This may be accomplished by surrounding the staff with a bottomless box or one having a few holes either in the bottom or in the sides that are parallel to the direction of flow.

79. Crest Height and End Clearances.—It was particularly pointed out by Bazin as well as by Fteley and Stearns that the height of the weir crest above the bottom of the channel was an influential factor in determining the value of the velocity of approach and the amount of crest contraction. Francis, as a result of his observations, believed it desirable that this height be at least equal to 3 H in order to obtain complete contraction. Since his coefficient was deduced from experiments where this condition was followed, it is essential that weirs to be gauged by his formulas should be so constructed.

In the formulas of Bazin and of Fteley and Stearns, provision was made for the varying of this dimension, but these engineers pointed out the desirability of keeping the water below the crest as deep as possible, since it reduced the velocity of approach and the probable error due to it. It has been shown experimentally

(and can be proven mathematically) that the effect of the velocity of approach is negligible if the area of the cross-section of the channel be about 6 times the product of bH. For a suppressed weir this would be the equivalent of requiring the height of the weir crest to be 5 times the head upon it. Generally this proportion it not found in practice and a velocity correction is necessary. In the case of a weir with end contractions all experimenters state that, for accurate application of their formulas, the end contractions must be made complete by allowing a clearance of at least 2 H and preferably 3 H between the ends of the weir and the sides of the channel.

If these clearances be observed, it can be shown that for values of b less than 10 H, the product of bH will be less than $\frac{1}{6}$ the area of the channel's section and the effect of the velocity of approach will be practically negligible.

80. Submerged Weir.—If the water level on the downstream side of a weir be made to stand above the level of the crest, the

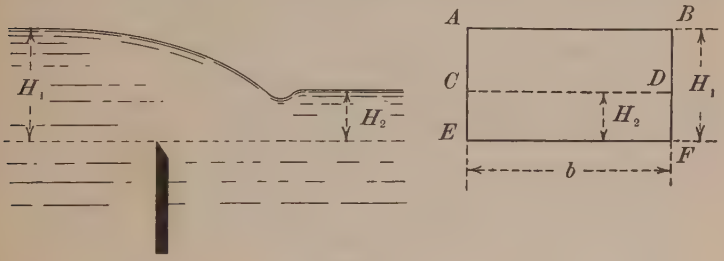

FIG. 76

weir is said to be *Submerged*. Using a weir with no end contractions, Fteley and Stearns experimented under the above conditions to determine a rational formula for flow and coefficients for the same. This formula may be obtained by considering that the flow takes place in two parts,—one through the weir $ABCD$ (Fig. 76), and the other through the orifice $CDEF$. The theoretical discharge through the weir part would be

$$Q = \tfrac{2}{3}\, b\, \sqrt{2g}\, (H_1 - H_2)^{\frac{3}{2}},$$

and through the orifice part,

$$Q = b\, H_2\, \sqrt{2g}\, (H_1 - H_2)^{\frac{1}{2}}.$$

By adding these, there results

$$Q = \frac{2}{3} \times \sqrt{2g}\left(H_1 + \frac{H_2}{2}\right)(H_1 - H_2)^{\frac{1}{2}}$$

Dubois and Stearns proposed

$$Q = d\left(H_1 + \frac{H_2}{2}\right)\sqrt{H_1 - H_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

d having a value dependent on $\frac{H_1}{H_2}$ as shown in the accompanying table:—

Constructions of the Weir Proper and Stearns Standard Weir Formula

$\frac{H_1}{H_2}$	0	1	2	3	4	5	6	7	8	9
0		3.33	3.33	3.34	3.34	3.35	3.37	3.37	3.37	3.37
0.1	3.37	3.38	3.38	3.39	3.39	3.40	3.42	3.43	3.43	3.20
0.2	3.20	3.18	3.17	3.16	3.16	3.15	3.14	3.13	2.13	3.12
0.3	3.12	3.11	3.10	3.10	3.10	3.10	3.10	3.17	3.17	3.16
0.4	3.10	3.11	3.11	3.11	3.11	3.11	3.11	3.12	3.12	3.12
0.5	3.11	3.11	3.11	3.10	3.10	3.10	3.10	3.10	3.10	3.10
0.6	3.09	3.09	3.09	3.09	3.09	3.09	3.09	3.09	3.09	3.09
0.7	3.09	3.09	3.09	3.09	3.09	3.09	3.10	3.11	3.11	3.12
0.8	3.13	3.13	3.13	3.14	3.14	3.15	3.16	3.16	3.17	3.18
0.9	3.19	3.20	3.21	3.22	3.23	3.25	3.26	3.28	3.30	3.33
1.0	3.30									

Experiments on submerged weirs are too few to give accuracy and certainty to the results obtained by formulas. It is difficult to determine the heads, especially the down-stream one. The fluctuation of the surface at this latter point makes refined measures out of the question. Such weirs should never be employed for accurate measurements. It may happen that, during the gauging of a natural stream, an unknown, rush of water may submerge the weir, and the above formula will then aid in securing a close approximation to the value of the discharge during the flood interval. Again, with channels having their water levels close to the top of their walls or banks, it may be impossible to construct a standard weir without raising the backwater sufficiently to overflow the sides. A submerged weir would then be a necessity

Bazin, in a description of his work on weirs offers a formula for the submerged weir and a reference to his work appears in the Literature at the end of this chapter.

81. Triangular Weir.—Triangular weirs are sometimes used when the quantity of water flowing is not large. The customary arrangement is shown in Fig. 77, both sides of the notch being equally inclined from the vertical. Inasmuch as the issuing stream is of similar cross-section for all heads, the value of the coefficient should be fairly constant. Experiment has shown this to be the case.

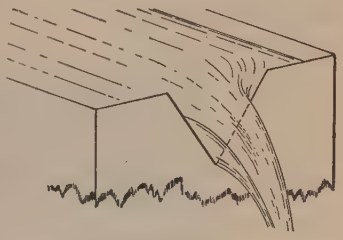

FIG. 77

The formula for theoretical discharge may be obtained as follows. In Fig. 78 let x be the head on an elementary horizontal strip. From similar triangles its length is $b(H - x) \div H$, and for its area we have

$$dA = \frac{b}{H}(H - x)dx.$$

Neglecting velocity of approach, the discharge through the strip (considered as an orifice) is

$$dQ = \frac{b}{H}(H - x)\, dx\sqrt{2gx} = \frac{b}{H}\sqrt{2g}\,(Hx^{\frac{1}{2}} - x^{\frac{3}{2}})dx,$$

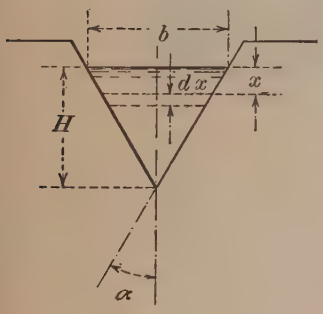

FIG. 78

and if this be integrated with H and 0 as the limits of x we obtain

$$Q = \tfrac{4}{15}\, b\sqrt{2g}\, H^{\frac{3}{2}} \quad . \quad (83)$$

The sides of the triangle being equally inclined, $b = 2H \tan \alpha$, and

$$Q = \tfrac{8}{15} \tan \alpha \sqrt{2g}\, H^{\frac{5}{2}}.$$

Introducing the usual coefficient to correct for friction and contraction,

$$Q = c.\tfrac{8}{15} \tan \alpha \sqrt{2g}\, H^{\frac{5}{2}}. \quad . \quad . \quad . \quad . \quad . \quad (84)$$

Little is known of the value of c as α and H vary. If the notch has a vertex angle of 90°, α is 45° and (84) becomes

$$Q = c.\tfrac{8}{15}\sqrt{2g}\,H^{\frac{5}{2}}.$$

We have two sets of reliable experiments from which to determine the value of c:—those of Professor James Thomson in 1861 and those of Mr. James Barr in 1908. Thomson used heads from 2 to

Discharge from Triangular Weir

7 inches in height and Barr extended these up to 10 inches. The results by both men were practically identical. It was found that the coefficient increased very slightly as the head decreased. Thomson estimated its mean value to be 0.593, which inserted in the above formula gives

$$Q = 2.536\,H^{\frac{5}{2}}. \qquad \ldots\ldots \quad (85)$$

Thomson's mean value of c varied about one-half per cent from the extremes and hence equation (85) has the same maximum

deviation from his experimental results. In Barr's experiments the channel of approach was 4 feet wide and the apex of the notch was 2 feet above the channel floor. It would be well to use these as minimum dimensions when employing the above formula, although the height of the apex may be made as small as 1 foot for heads less then 8 or 9 inches.

Neither (84) nor (85) provide for velocity of approach and if the latter is present to an appreciable degree, H should be replaced by $(H + 1.4\ h)$, where $h = \frac{v_o^2}{2g}$, v_o being the average velocity of approach.

Mr. A. A. Barnes, from a study of Thomson's and Barr's work proposes

$$Q = 2.48\ H^{2.48} \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

as fitting all of Barr's experimental runs with an error less than one-fifth per cent. In view of the fact that the constant of equation (85) is only a *mean* value, it appears that Barnes' empirical formula is to be preferred. Barnes recommends adding to H a value, $0.01\ v_o^2$, whenever the mean velocity of approach is larger than 0.50 feet per second. It is probable that Barnes' formula can be used for heads considerably greater than 10 inches without increasing the error. Further investigation must determine its limitations.

82. Trapezoidal Weir of Cipolletti.—This weir, invented by an Italian engineer whose name it bears, is mentioned because of its ingenious design and the fact that it is more or less used in irrigation work in this country. As the name indicates, the notch is of trapezoidal form as shown in Fig. 79. The side slopes are alike and have an inclination of 1 horizontal to 4 vertical. The reason for this is of interest. The discharge may be considered in two parts,—one through the rectangular area of length b, and the other through a triangular area having a base width of 2 d. The total dis-

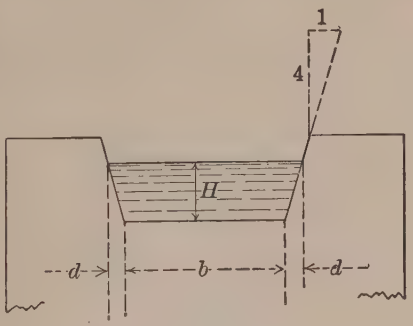

FIG. 79

charge is, therefore, greater than from a rectangular *contracted* weir of length b. Cipolletti proposed giving the sides such a slope that this increase would be just equal to the decrease in discharge through a contracted weir, caused by end contractions. This would make the trapezoidal weir the equivalent of a rectangular *suppressed* weir of length b. The increase, being the discharge through the end triangles, may be written from equation (83), as

$$Q = c \times \tfrac{4}{15} 2 d \sqrt{2 g}.H^{\frac{3}{2}},$$

and the decrease due to end contractions is, according to Francis,

$$Q_1 = c \times \tfrac{2}{3} \sqrt{2 g} \times 0.2 H^{\frac{5}{2}}. \quad \text{(See equation 65.)}$$

Equating these values and assuming the two values of c to be alike, there results

$$d = \frac{H}{4},$$

giving the slope which Cipolletti recommended.

From his own experiments and those of Francis, Cipolletti proposed

$$Q = 3.367\, bH^{\frac{3}{2}}. \quad . \quad . \quad . \quad . \quad . \quad (87)$$

the correction for velocity of approach to be made as in the formula of Francis (Art. 71).

Messrs. Flinn and Dyer in 1893 experimented with a weir of this type and obtained a value for the coefficient of 3.283. This was the mean value derived from 32 experiments, in which the head varied from 0.3 to 1.25 feet. In correcting for the velocity of approach, however, they followed the method of Hamilton Smith (total head $= H + 1.4\, h$), while Cipolletti used the method of Francis. (See Art. 71.) The difference in method would nearly account for the difference in coefficient; and their conclusion was that the formula proposed by Cipolletti himself gave results within one per cent. of the truth.

The chief advantage of this weir lies in the fact that it permits the use of a contracted weir where a suppressed weir would be impracticable, yet avoids the tedious computations involved in correcting b for end contractions by the Francis method. It is

often used in measuring water in irrigation ditches where there are many varying heads to be recorded and computed. Its use is then a matter of convenience, and its accuracy is sufficient for the purpose.

83. Measurement by Weirs.—Both the rectangular and 90° triangular weir offer means for accurately measuring the flow of streams, provided certain precautions, some of which have been enumerated, are taken. The triangular notch is somewhat limited in capacity as little is known of its coefficient above heads of about 1 foot. The rectangular weir is commonly used, preference being given to the *suppressed* type because of the absence of any uncertainty due to the effect of end contractions. The following conditions should be closely adhered to, in the case of either the contracted or suppressed weir, if results having a good degree of accuracy are sought.

(1) The crest must be sharp-edged and horizontal; and its length should be greater than 3 H, especially for a contracted weir.

(2) The height of the crest above the bottom of the channel should be made large enough to reduce the velocity of approach to a negligible value. In a suppressed weir this requires that the crest height be equal to or greater than 5 H. For the contracted weir, this will result if the depth of water back of the weir be sufficient so that the sectional area, A, of the channel will be equal to or greater than 6 bH. The crest height should always be sufficient to insure complete contraction on the under side of the nappe. All investigators insist that the crest height be at least 2 H and preferably 3 H.

(3) End contractions, when present, should be made complete by observing a clearance of 3 H between the ends of the weir and the sides of the channel.

(4) Provision must be made at the ends of the weir for the free admission of air to the space beneath the falling sheet or *nappe*. The existence of a partial vacuum here causes a depression of the *nappe* and Bazin's experiments show that an increased discharge follows.

(5) The head must not be so low as to permit the *nappe* clinging to the front face of the weir.

(6) The plane of the weir's upstream face must be vertical. An upstream inclination *reduces*, and a downstream inclination

increases the discharge (Fig. 80). Bazin, in an account of his experiments, gives a discussion on the effect of inclination and also comparative values for the discharge as the inclination varies.

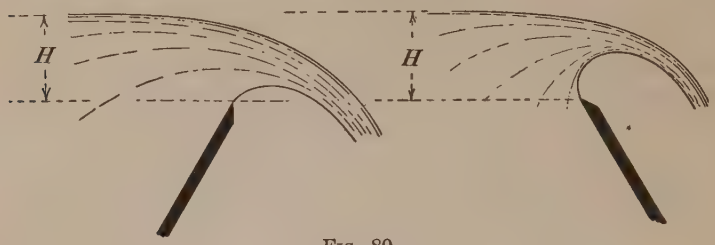

FIG. 80

(7) The head H must be accurately determined. This is best done by using the hook gauge (described in Art. 55) in connection with a measuring box or pail as described in Art. 78. It is recommended that, where possible, the measurement be made at a distance 6 feet upstream from the crest, although Bazin made his distance 16.4 feet. When it is not possible to observe this distance, care should be taken to make it sufficient to escape the effect of the surface curve.

The measuring box (Art. 78) should always be used as Mills has conclusively shown that when constructed properly its level will be truly that of the water in the channel at the point where the connecting tube is inserted. The use of a pipe, with perforations in its top or bottom sides, extending across the bottom of the channel, should never be attempted. Professor Williams at Cornell demonstrated that very large errors might result from its use.

(8) When the weir is constructed at the end of a large box into which water is being delivered, the box should be long enough to allow the flow to become steady and free from large eddies before meeting the crest. Gibson believes that this distance should be at least 30 H and it is desirable in every case that baffles or screens be constructed across the head of the channel in order to quiet the water as much as possible. Vertical or horizontal slats in the form of a grid, or even boards containing many holes may be used. Wide slats placed close together and having their faces parallel to the flow give the best results.

If all the above precautions be observed, and that formula selected which is adapted to the weir (Art. 77), it is possible that the

results obtained may be in error less than 2 per cent. Such accuracy is not to be expected, however, save only by an experienced observer.

84. Weirs with Rounded or Wide Crests.—Messrs. Fteley and Stearns experimented with weirs having the upstream edge

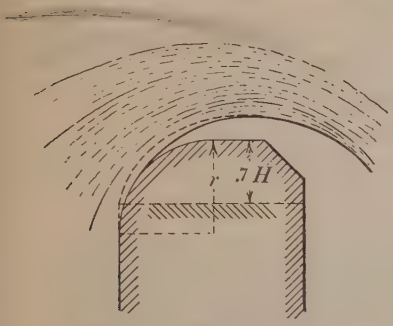

Fig. 81

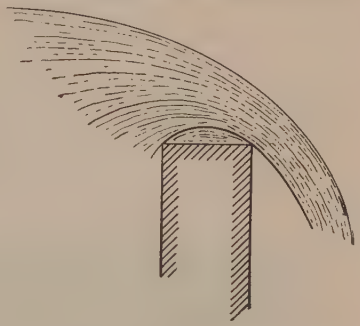

Fig. 82

of the crest slightly rounded. They found that the rounding increased the discharge by diminishing the contraction. The case is comparable with that of the orifice. When the rounding was made by a quarter circle having a radius of less than 0.5 inch, they found that the discharge could be determined by the usual formula, provided 0.7 the radius (in feet) be added to the measured head (Fig. 81).

If the crest be squared, but made so wide as to cause the water to touch again after leaving the inner edge, the amount discharged is usually diminished by the increase in friction. The experiments of Fteley and Stearns illustrate two possible cases. Figure 82 shows a crest just wide enough to slightly interfere with the descending sheet. The flow may be increased slightly if the space between the crest and the water contains a partial vacuum. If the crest be so wide as to resemble Fig. 83, the flow will be diminished by the increase in frictional resistance. Rounded or wide crests should be avoided in precise measurements. For formulas approximating their discharge, the reader

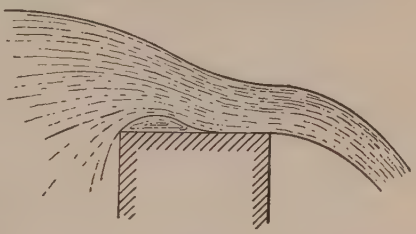

Fig. 83

is referred to "Hydraulics and Its Applications" by Gibson. (See Literature at end of Chapter.)

85. Dams used as Weirs.—The many dams existing on natural streams offer opportunities for gauging the flow if their coefficients, M, for use in the general formula,

$$Q = M\ b\ H^{\frac{3}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

can be ascertained. Since the shapes or profiles of dams vary greatly in detail, it would be a difficult and laborious task to determine and tabulate the coefficients for the many various types. It would have to be accomplished mainly by the use of models and the assumption made that the values of M found for the models

A Dam which Might Serve as a Weir. (Courtesy Continental Pipe Mfg. Co.)

would hold good for their prototypes. Were it possible to duplicate *exactly* in the model all the conditions present in the dam, the law of hydraulic similitude shows that the same coefficient would apply to both.

Two excellent studies, wherein models of dams were used, have been made. The first was by Bazin in 1897 and the second, in 1898, was carried out by G. W. Rafter at the Cornell Hydraulic Laboratory for the U. S. Deep Waterways Commission. The results of both experiments appear in a paper by Rafter found in the Transactions of the American Society of Civil Engineers, Vol. 44, 1900. Values of M are given for a great variety of profiles and conditions.

86. Discharge by Weirs under a Falling Head.—Let it be required to compute the time that would have to elapse while

reducing by a certain amount, the head on a given weir supplied by a reservoir of constant area, A. The problem is very similar to that of the orifice worked out in Art. 57.

From the weir in dt second of time, a small quantity

$$dQ = c\,b\,H^{\frac{3}{2}}dt,$$

would be discharged, causing a decrease, dH, in the head. Evidently,

$$dQ = c\,b\,H^{\frac{3}{2}}.dt = -A\,d\,H$$

and

$$t = \int_o^t dt = -\int_{H_1}^{H_2} \frac{A\,dH}{cbH^{\frac{3}{2}}} = \frac{2\,A}{cb}\left(\frac{1}{\sqrt{H_2}} - \frac{1}{\sqrt{H_1}}\right). \quad . \quad (89)$$

It will be noted that if H_2 be given the value of zero, the value of t will be found to be *infinity* and the problem is indeterminate. The explanation is that *both* the *head* and the *area of discharge* are approaching zero as a limit. Were the area *fixed*, a definite time would result. Francis' value of 3.33 may be used for c if the weir be a *suppressed* weir, while for a *contracted* weir the value obtained from the Smith formula will be found to simplify the computation.

87. Effect on Computed Discharge from Error in Measuring Head.—The question may arise as to what extent an error in measuring the head will affect the computed discharge. For the rectangular and triangular weirs this may be computed as follows:

(*a*) *Rectangular Weir.*

$$\text{Since } Q = KH^{\frac{3}{2}},\ \frac{dQ}{dH} = 1.5\,KH^{\frac{1}{2}}$$

$$\text{or } dQ = 1.5\,KH^{\frac{1}{2}}dH.$$

$$\text{Then } \frac{dQ}{Q} = \frac{1.5\,KH^{\frac{1}{2}}.dH}{KH^{\frac{3}{2}}} = 1.5\,\frac{dH}{H}$$

That is, a percentage change in H produces 1.5 times the same percentage change in Q, or an error of 1 per cent. in measuring H will produce 1.5 per cent. error in the computed Q.

(*b*) *Triangular Weir.*—
Proceeding as above,

$$Q = KH^{\frac{5}{2}}, \quad \frac{dQ}{dH} = 2.5\, KH^{\frac{3}{2}}$$

$$\text{or } dQ = 2.5\, KH^{\frac{3}{2}}\, dH.$$

$$\text{Then } \frac{dQ}{Q} = \frac{2.5\, KH^{\frac{3}{2}}\, dH}{KH^{\frac{5}{2}}} = \frac{2.5\, dH}{H}$$

Therefore a small change or error of 1 per cent. in the head produces an error of 2.5 per cent. in the computed discharge.

PROBLEMS

1. A rectangular weir with end contractions has a crest 10.37 ft. long and 2.87 ft. above the bottom of the channel. If the channel width be 16 ft., what amount of water will be discharged under a head of 0.875 ft.? Use Francis' formula. What depth will be found back of the weir?

2. A suppressed weir having a crest length of 6.80 ft., a height of 2.5 ft., discharges under a head of 0.67 ft. Compute the rate of discharge and the water depth back of the weir. Use Francis' formula.

3. A contracted weir is to be built in a rectangular channel, 10 ft. wide, discharging a quantity of 8 cu. ft. per second. What length and crest height should the weir have in order that the head shall not exceed 8 inches or the water depth behind the weir 3 ft.? Use Francis' formula.

4. Solve Problem 2 by use of the Bazin, Stearns, and Barnes formulas.

5. A rectangular channel 15 ft. wide contains water flowing 4 ft. deep with a mean velocity of 2.2 ft. per second. If a suppressed weir, 4.5 ft. high, be built across the channel, how much will the level of the water back of it be raised? Use Bazin's formula.

6. A rectangular flume, 10 ft. wide and 6 ft. deep, is carrying water to a depth of 4 ft. with a mean velocity of 1.5 ft. per second. How high above the bottom of the channel may the crest of a suppressed weir, 10 ft. long, be placed and not overflow the sides of the flume?

7. A suppressed weir, 6.97 ft. long, has its crest 2.79 ft. above the bottom of the channel. Compute the discharge under a head of 0.679 ft., using (*a*) Fteley and Stearns' formula; (*b*) Bazin's formula; (*c*) Barnes' formula.

8. A rectangular channel has a maximum discharge of 160 cu. ft. per second when the water flows 4 ft. deep. If the width be 20 ft., how high should the crest of a submerged weir be in order that the water back of the weir shall not be over 5 ft. deep when the weir is built and discharging the given maximum flow? *Ans.* 2.95 ft.

9. Given a submerged weir 9 ft. long with the upstream head 2.6 ft. and the downstream head 1.22 ft., compute the probable discharge. *Ans.* 106 sec.-ft.

10. A triangular weir has a 90 degree notch. What head will be necessary to discharge 1000 gallons per minute? Use Barnes' formula.

11. A triangular weir has a 60 degree notch. Compute the discharge under a head of 1.6 ft., assuming a coefficient of 0.64.

12. A triangular weir has one side sloping at 45° and the other at x horizontal to 1 vertical. Assuming the coefficient of discharge to be 0.62, what value of x should give a discharge of 10 cu. ft. per second under a head of 1 ft.?

13. A triangular weir has one side sloping at 45° and the other has a slope of 6 horizontal to 2 vertical. Assuming the coefficient of discharge as 0.62, find the probable rate of discharge under a head of 1.44 ft.

14. A Cipolletti weir with a crest length of 5.87 ft. discharges under a head of 0.875 ft. Compute the rate of discharge.

15. Compute the *theoretical* rate of discharge over a trapezoidal weir 2 ft. long on the crest, and having one side vertical and the other sloping outward 2 horizontal to 1 vertical, under a head of 2 ft.

16. Compute the rate of discharge over a trapezoidal weir, crest length 5 ft., ends sloping out at 45°, head 1 ft. and velocity of approach negligible. Assume coefficient of discharge 0.62.

17. A reservoir, 500,000 sq. ft. in area, is to be controlled by a concrete spillway with a permanent crest at Elev. 100 ft. It is intended to make the length of the spillway such that (by removing flash-boards) water can be drawn from Elev. 104 down to Elev. 102 in 30 minutes of time. What should be the length of the spillway if $M = 3.5$?

18. A reservoir whose area is 12,000 sq. ft. has an outlet through a contracted weir whose crest is 3 ft. long. How long a time will be required to lower the reservoir level 1 ft. from an initial head of 1.60 ft. on the weir crest? Use Hamilton Smith's formula and a mean value for the coefficient. *Ans.* 20.9 min.

19. A reservoir, 50 ft. by 200 ft. in plan, has its sides vertical. It discharges through a rectangular suppressed weir, the initial head being 15 in. How long is the crest if 30 minutes are required to lower the level 13.50 in.?

Ans. 6.47 ft.

20. A reservoir 150 ft. by 200 ft. in plan, discharges over a suppressed weir 10 ft. long. Beginning with a head of 2 ft., how long a time will be necessary to lower the water level 18 inches?

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CHAPTER VIII

FLOW THROUGH PIPES

88. The use of the term *pipe* is generally limited, in its application, to closed conduits carrying water *under pressure.* Pipes are commonly circular in section, this form combining the advantage of structural strength with that of simple construction. The circular pipe also has the following important property. Of all pipes having equal sectional areas, but differing in *shape* of section, the circular pipe has the smallest *perimeter* of section and, therefore, per foot of length, the smallest inside wall area. It follows that the resistance which a circular pipe offers to water flowing through it, is less than in a pipe of any other section.

When pipes carry water with their sections only partially filled, as in the case of sewers and large aqueducts, it is sometimes of advantage to employ sections other than circular. Such conduits, being under no pressure, do not carry the flow by reason of an external head which they may be under, but depend upon the inclination of the conduit and the surface of the water, to give to the latter its velocity. They are of the nature of *open channels* and, although they obey the same fundamental laws as do the *pressure* pipes, it will be found more convenient to classify them with *open channels* and discuss them in the following chapter. Evidently a pipe could be so constructed and laid that it would flow completely full and yet have the water exerting no upward pressure against its top. As in the previous case, velocity of flow would depend upon the inclination of the pipe and the conditions would simulate those of open channel flow.

In view of the above facts, the present discussion will be confined to pipes of circular section; but at the proper point it will be shown how the formulas derived may be used, with slight modification, for sections of any shape.

89. Straight Pipe, Uniform Diameter.—A very simple case of flow occurs when a fairly straight pipe of uniform section is inserted in the side of a reservoir and allowed to discharge under a constant

head into the atmosphere. (Fig. 84.) The flow being steady we may write Bernoulli's Equation between points m and n, obtaining,

$$h = \frac{v^2}{2g} + \text{Lost Head.} \quad . \quad . \quad . \quad . \quad (90)$$

The last term covers all the losses occurring between the two points, no matter how caused. In the assumed case, the only losses present are:

(a) Head lost at entrance to the pipe.

(b) Head lost by internal friction between the water particles.

(c) Head lost by friction between pipe walls and the moving water.

If each of these can be expressed as a function of the velocity in the pipe, then equation (90) will serve to determine the velocity

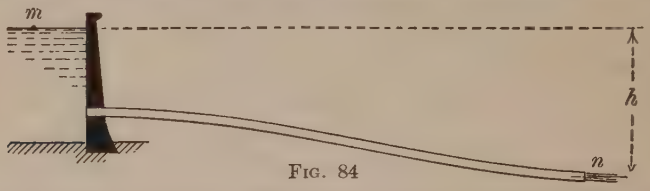


FIG. 84

when the head h is known. The last two of the above losses are so inextricably involved, one in the other, that no attempt will be made to separate them. They will be jointly studied as a loss by *Pipe Friction.*

(a) Entrance Loss. If the end of the pipe entering the reservoir is flush with the inner wall surface, the conditions of flow in the first three diameters of the pipe's length are identical with those existing in a standard short tube as described in Art. 61. There is the same contracting of the stream after passing the entrance edge of the pipe, and the same subsequent expanding to fill the whole section of the pipe. It is perfectly logical, therefore, to reason that the losses in head in the two cases are identical in nature and may be expressed by like equations. In Art. 61 it was shown that the head lost in the short tube might be expressed as one-half of the velocity head found in the full section. Accordingly, for the first three diameters of the pipe we may write,

$$\text{Entrance Loss} = 0.5\,\frac{v^2}{2g}. \quad . \quad . \quad . \quad . \quad . \quad (91)$$

This value has been proved substantially true by experiments.

If the pipe be made to project a short distance into the reservoir, conditions then simulate those discussed in Art. 68, and

$$\text{Lost Head} = 0.78\frac{v^2}{2g}$$

If the entrance to the pipe be made bell-mouthed or rounded, the loss is much reduced and, as explained in Art. 66, may be computed from

$$\text{Lost Head} = 0.15\frac{v^2}{2g}$$

(b) Pipe Friction Loss. Probably no other subject in the field of analytical and experimental hydraulics has received the attention which has been given to the study of the laws which control the loss due to pipe friction. Thousands of experiments have been made without resulting in conclusive evidence of the exact relations which exist between this loss and the various physical factors affecting it. We have learned, however, certain definite facts regarding it which may be briefly summarized as follows:

The resistance offered to flow is

(a) *Independent of the pressure in the pipe.*

(b) Proportional to the extent of the frictional surface of the pipe.

(c) Variable with the velocity of flow so that for any one pipe we may write,

Frictional resistance $= k\,v^n$.

The value of n is unity when the velocity is below the *critical.* (See Art. 91.) For velocities above the critical its value depends on the temperature of the water and the character of the pipe's inner surface.

It will at once be seen how difficult and probably impossible it is to express these facts in the form of a rational mathematical equation. The best we can do at present is to formulate an expression having a *rational basis* and depend upon experiments to furnish *empirical* values for certain constants and exponents which are involved. One treatment of the problem is as follows:

Fig. 85 represents a portion of the pipe shown in Fig. 84. We will assume that the resistance to flow is sufficient in the long pipe to cause the pipe to flow full in its entire length. It will be true,

then, that the average value of v, will be the same for all sections of the pipe, since

$$av_m = av_n.$$

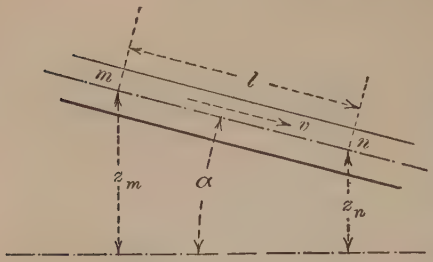


FIG. 85

Across any one section the velocity of course varies having a maximum value at or near the center and a minimum near the sides of the pipe. We will neglect this variance and consider all particles to have a mean velocity v. Referring to Fig. 85, m and n being two points on the pipe's axis, we may write,

$$\frac{v_m^2}{2g} + \frac{p_m}{w} + z_m = \frac{v_n^2}{2g} + \frac{p_n}{w} + z_n + \text{Lost Head},$$

and since the velocities are equal (the pipes being of uniform section),

$$\text{Lost Head} = \left(\frac{p_m}{w} + z_m\right) - \left(\frac{p_n}{w} + z_n\right). \quad . \quad . \quad (92)$$

To otherwise express the loss in head, consider the cylinder of water between sections m and n, and the forces acting on it (Fig. 86). ap_m and ap_n are the pressures exerted on the ends by the adjacent particles, a representing the area of cross-section. R is the total frictional resisting force, and awl is the weight of the water. The motion being uniform, we may write

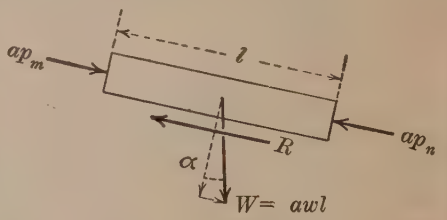


FIG. 86

$$ap_m + awl.\sin\alpha - R - ap_n = 0,$$

or

$$\frac{p_m}{w} + l\sin\alpha - \frac{p_n}{w} = \frac{R}{aw}.$$

From Fig. 85 we have

$$l\sin\alpha = z_m - z_n,$$

which inserted in the above gives

$$\left(\frac{p_m}{w} + z_m\right) - \left(\frac{p_n}{w} + z_n\right) = \frac{R}{aw},$$

and this combined with (92) results in

$$\text{Lost Head} = \frac{R}{aw}.$$

If now the frictional resistance per unit area of pipe surface be F, so that $R = \pi dlF$, we may write

$$\text{Lost Head} = \frac{4\pi d\, lF}{w\pi d^2} = \frac{4\, lF}{wd}. \quad . \quad . \quad . \quad (93)$$

As to the value of F, it was pointed out in the first part of the present paragraph that exact knowledge of its magnitude, under varying conditions of surface roughness, temperature, velocity and diameter is at present lacking. *Rational* treatment, therefore, ceases at this point and experimental knowledge must be relied upon to carry us farther. It would appear that we might substitute for F a value of $k\, v^n$, assuming that the value of k would be constant for any one pipe, varying only as the roughness of the inner surface varied. But the loss which we are attempting to evaluate comprises loss by *internal* friction as well as by wall friction, and experiments show that the former varies slightly with water temperature and the diameter of the pipe. The direct effect of temperature is a change in viscosity, but it is found that for turbulent or sinuous flow (Art. 91) the temperature effect may be neglected since it is relatively very small. As a fact worthy of note, water changes its viscosity nearly 600 per cent. through a temperature range from near freezing to boiling points.

The effect of diameter changes upon internal resistance is found to be inversely proportional to a power of the diameter slightly greater than the first. We may, therefore, write

$$F = \frac{kv^n}{d^m}$$

as expressing the above experimentally determined facts. Substituting this value in (93), we obtain

$$\text{Lost Head} = \frac{4\, k}{w} \cdot \frac{lv^n}{d^{m+1}}$$

or more simply,

$$h_f = K\frac{lv^n}{d^x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

in which K is a constant whose value depends only on *surface roughness* (temperature neglected). Practically all investigators agree upon (94) as expressing the law of resistance, and the many formulas for pipe flow may be shown to have this equation for their basis.

To make the equation usable, numerical values must be found for K, n and x. Experiments have shown n to have values ranging from about 1.70 for very smooth pipes to slightly more than 2 for pipes with rough interiors. The value of x similarly varies with roughness and has approximate limits of 1.10 and 1.40. For a *given pipe*, K has been found to be approximately constant when n and x are given their proper values, but varies with velocity and diameter if n and x are assigned other values. Between pipes of the same material and diameter a slight discordancy in the values of K, n and x may result because two pipes of the same material may vary somewhat in their roughness. In general, two methods have been followed in evaluating equation (94).

(a) Assume constant values of n and x, irrespective of pipe material and determine by experiment proper values of K to accord with different values of material, diameter and velocity.

(b) Determine experimentally the proper value of K, n and x for each kind of pipe material.

A very simple formula may be obtained by assuming n to have a value of 2, and x a value of unity. This was first proposed by Darcy in 1857 and later Weisbach introduced the constant $2g$ giving

$$\text{Lost Head} = f\frac{l}{d}\cdot\frac{v^2}{2g} \quad . \quad . \quad . \quad . \quad . \quad (95)$$

The introduction of $2g$ not only results in allowing the loss to be expressed as a function of the *velocity head*, but it makes the coefficient f an abstract number, independent, for its value, of the system of units of measure employed. This may be seen from the following interpretation of (95).

$$\text{Distance} = f \times \frac{\text{distance}}{\text{distance}} \times \frac{\dfrac{\text{distance}^2}{\text{time}^2}}{\dfrac{\text{distance}}{\text{time}^2}}$$

or

$$f = \text{an abstract number.}$$

This formula has for years been widely used in preference to others, owing to its simplicity. It is commonly called the formula of Chezy although it is different in form from that proposed by him. (See Art. 104.) It is gradually being superceded by others having exponents n and x more nearly representative of the facts, but in view of the long service it has rendered and its familiarity to engineers, its use by many will continue.

We may now revert to the discussion of flow in a straight pipe of uniform diameter, begun in the first of this article, and amplify the last term of equation (90) by substituting expressions for head lost at entrance and by pipe friction.

There results,

$$h = \frac{v^2}{2g} + \frac{.5\,v^2}{2g} + \frac{fl}{d}\,\frac{v^2}{2g}. \quad . \quad . \quad . \quad . \quad (96)$$

as expressing the relation between head and velocity in the case of a fairly straight pipe of uniform diameter.

90. Determination of f in Chezy's Formula.—The quantity f is generally spoken of as the *Friction factor* and its value for any particular pipe and velocity of flow may be obtained experimentally as follows:—

Figure 87 shows two piezometers inserted in a pipe at two sections a distance l apart. Since the velocity at both sections is the same (diameter constant), we may write,

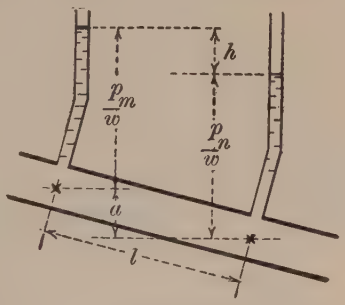


FIG. 87

$$\frac{p_m}{w} + a = \frac{p_n}{w} + \text{Lost Head},$$

and

$$\text{Lost Head} = \frac{p_m}{w} - \frac{p_n}{w} + a = h.$$

By measuring h, a, and Q the value of the lost head is found, and (95) may be solved for f.

As a result of many experiments it has been found that

(1) f decreases, in the same pipe, as velocity increases.

(2) f decreases, with the same velocity, as the diameter increases.

(3) f varies *largely* with the condition of the interior surface of the pipe, being larger, the rougher the surface.

Facts (1) and (2) agree with previous theoretical considerations. We should expect f to vary with velocity and diameter since the values given to the exponents of these quantities are only approximately correct. Were they *quite* correct, f would remain constant for any one pipe so long as the temperature of the water were constant.

Values of f have been determined for a number of materials, including cast-iron, wood, lead, brass, glass, wrought-iron and riveted steel. When values obtained by different investigators from pipes of a given material are compared, they are often found to be discordant probably because of slight differences in the physical conditions of the pipe used and errors in measuring the head lost. Not only will the same material vary in the roughness of its finish, but experiments made upon long pipes, under service conditions, where joints are frequent and slight inaccuracies in alignment are present, should give slightly different results than would be obtained by laboratory experiments upon short lengths of pipe having few or no joints and losses so small in magnitude as to make their exact measurement difficult.

From a review of the most trustworthy experiments of the former class, the writer has compiled the following tables giving values of f for clean, smooth cast-iron pipe, small wrought-iron pipe, fire hose and wood stave pipe. Those for iron pipe are based on the Williams-Hazen formula for pipe flow (Art. 105) while those for stave pipe are based on studies of the experiments of Morritz, Scobey, Moore and Noble on this kind of pipe. The values for fire hose are those obtained by Merriman from a study of Freeman's well known experiments, which were made at Lawrence, Mass., in 1888.

In all the tables, the figures given represent average values only, and, therefore, must be considered as approximate. Errors of 10 per cent. or more may doubtless result in some cases from their use. If the giving of each value to three significant figures be questioned, the writer would simply point out the fact that they were taken, in most cases, from plotted curves and accurately represent the variation in average values.

VALUES OF f FOR CLEAN, SMOOTH CAST-IRON PIPES

DIAM. IN INCHES	VELOCITY IN FEET									
	1	2	3	4	5	6	8	10	15	20
4	.0285	.0255	.0240	.0230	.0225	.0220	.0210	.0200	.0190	.0180
5	.0275	.0245	.0230	.0225	.0215	.0210	.0200	.0195	.0185	.0175
6	.0265	.0240	.0225	.0215	.0210	.0200	.0195	.0190	.0175	.0170
8	.0255	.0230	.0215	.0205	.0200	.0195	.0185	.0180	.0170	.0160
10	.0245	.0220	.0205	.0200	.0190	.0185	.0180	.0175	.0165	.0155
12	.0235	.0215	.0200	.0190	.0185	.0180	.0175	.0170	.0160	.0150
15	.0230	.0205	.0195	.0185	.0180	.0175	.0170	.0165	.0155	.0145
18	.0220	.0200	.0190	.0180	.0175	.0170	.0165	.0160	.0150	.0140
20	.0215	.0195	.0185	.0175	.0170	.0165	.0160	.0155	.0145	.0140
24	.0210	.0190	.0180	.0170	.0165	.0160	.0155	.0150	.0140	.0135
30	.0200	.0185	.0175	.0165	.0160	.0155	.0150	.0145	.0135	.0130
36	.0195	.0180	.0170	.0160	.0155	.0150	.0145	.0140	.0130	.0125
42	.0190	.0175	.0165	.0155	.0150	.0145	.0140	.0135	.0130	.0125
48	.0185	.0170	.0160	.0155	.0150	.0145	.0140	.0135	.0125	.0120
60	.0180	.0165	.0155	.0150	.0145	.0140	.0135	.0130	.0120	.0115
72	.0175	.0160	.0150	.0145	.0140	.0135	.0130	.0125	.0120	.0115
84	.0170	.0155	.0145	.0140	.0135	.0130	.0125	.0120	.0115	.0110
96	.0165	.0150	.0140	.0135	.0130	.0125	.0120	.0120	.0110	.0105

VALUES OF f FOR SMALL WROUGHT-IRON PIPES

NOMINAL DIAM. IN INCHES	ACTUAL DIAM. IN INCHES	VELOCITY IN FEET									
		1	2	3	4	5	6	8	10	15	20
¾	0.824	.0430	.0390	.0365	.0350	.0340	.0330	.0320	.0305	.0290	.0280
1	1.048	.0415	.0370	.0350	.0335	.0325	.0315	.0305	.0295	.0275	.0265
1¼	1.38	.0395	.0355	.0335	.0320	.0310	.0300	.0290	.0280	.0265	.0255
1½	1.61	.0385	.0345	.0325	.0315	.0300	.0295	.0280	.0275	.0255	.0245
2	2.0	.0370	.0335	.0315	.0300	.0290	.0285	.0270	.0265	.0245	.0235
2½	2.5	.0355	.0325	.0305	.0290	.0280	.0275	.0260	.0255	.0240	.0230
3	3.0	.0345	.0315	.0295	.0280	.0270	.0265	.0255	.0245	.0230	.0220

VALUES OF f FOR 2½-INCH FIRE HOSE

DESCRIPTION OF HOSE	VELOCITY IN FEET				
	4	6	10	15	20
Unlined canvas	.038	.038	.037	.035	.034
Rough rubber-lined cotton	.032	.031	.031	.030	.029
Smooth rubber-lined cotton	.024	.023	.022	.019	.018

VALUES OF f FOR WOOD PIPE

(a) Very Smooth,—Planed. (b) Smooth,—Unplaned

DIAM. IN INCHES		1	2	3	4	5	6	8	10	15	20
12	*a*	.0253	.0207	.0183	.0168	.0157	.0149	.0137	.0128	.0113	.0103
	b	.0367	.0300	.0265	.0243	.0228	.0216	.0198	.0186	.0164	.0150
24	*a*	.0231	.0188	.0167	.0153	.0143	.0137	.0126	.0117	.0103	.0095
	b	.0335	.0273	.0242	.0222	.0208	.0198	.0182	.0170	.0150	.0138
36	*a*	.0220	.0179	.0159	.0146	.0137	.0130	.0119	.0112	.0099	.0090
	b	.0318	.0260	.0230	.0212	.0198	.0188	.0173	.0162	.0144	.0132
48	*a*	.0212	.0172	.0153	.0141	.0132	.0125	.0115	.0108	.0095	.0088
	b	.0307	.0250	.0222	.0204	.0191	.0181	.0166	.0156	.0138	.0127
60	*a*	.0206	.0168	.0149	.0137	.0128	.0122	.0112	.0105	.0092	.0085
	b	.0299	.0244	.0216	.0199	.0186	.0176	.0162	.0152	.0134	.0124
72	*a*	.0202	.0164	.0145	.0134	.0125	.0118	.0109	.0102	.0090	.0083
	b	.0292	.0238	.0210	.0194	.0181	.0172	.0158	.0148	.0131	.0120
84	*a*	.0197	.0160	.0142	.0131	.0123	.0116	.0107	.0100	.0088	.0081
	b	.0286	.0232	.0206	.0190	.0178	.0168	.0155	.0145	.0128	.0118
96	*a*	.0194	.0159	.0141	.0129	.0121	.0115	.0106	.0099	.0088	.0080
	b	.0281	.0230	.0204	.0187	.0175	.0166	.0153	.0143	.0127	.0116
108	*a*	.0191	.0157	.0139	.0127	.0119	.0113	.0103	.0097	.0086	.0079
	b	.0277	.0227	.0201	.0184	.0173	.0164	.0150	.0141	.0125	.0115
120	*a*	.0189	.0155	.0137	.0126	.0117	.0112	.0102	.0096	.0085	.0078
	b	.0274	.0224	.0198	.0182	.0170	.0162	.0148	.0139	.0123	.0113
132	*a*	.0186	.0152	.0135	.0124	.0116	.0110	.0101	.0095	.0085	.0077
	b	.0270	.0221	.0196	.0180	.0168	.0160	.0146	.0137	.0122	.0112
144	*a*	.0184	.0150	.0134	.0123	.0115	.0109	.0100	.0094	.0083	.0076
	b	.0267	.0218	.0194	.0178	.0166	.0158	.0145	.0136	.0120	.0110

Experiments made upon iron pipes which have been in use for a long time, show values of f in general much greater than given in the tables. This is because of the gradual roughening of the pipe due to the accumulation of rust tubercles or other matter upon the sides of the pipe. This deterioration of the pipe with age follows no rule, as the rate is dependent upon the quality of the water, the quality of the metal and largely upon the durability of the paint preservative with which it is commonly the custom to cover all iron pipes. Asphaltum and coal tar have been much used for this latter purpose. In designing pipes of a desired capacity, it is evidently the *future* value of f that is important since the roughening will be accomplished in a comparatively few years. A rule sometimes followed is that of doubling the value of f, as given in the tables, to allow for deterioration and while this method serves to

fix in mind the fact that deterioration is rapid and considerable, it does not at all accurately express the rate of its progress. Pipes of small diameter deteriorate in carrying capacity faster than do large pipes, owing to the proportionately greater effect of wall resistance and to the fact that the section area is rapidly reduced by incrustations. From a study of observations made on old pipes whose sizes ranged from 12 to 60 inches, the writer has found that very good results may often be obtained if the value of f as given in the preceding table for cast-iron pipes be multipled by an *age factor* whose value varies with the age of the pipe as follows:

Age of Pipe, Years	Age Factor
0	1.00
5	1.15
10	1.35
15	1.50
20	1.75
25	1.90
30	2.10
40	2.60
50	3.00

For small pipes the value of f increases more rapidly than the above figures would indicate. For the purpose of comparison, the following table is given showing the lost head in several old pipes as measured by the investigator and as computed by the above method of age factors. The data is from Hughes and Safford's "Hydraulics."

Diam. Inches	Age, Yrs.	Length	Velocity	Head lost per 1000 ft.		Investigator
				Measured	Computed	
60	8	8215	1	0.082	0.07	Fenkell
48	17	1747	2	0.333	0.39	Fitzgerald
48	17	1747	5	2.12	2.18	"
36	3	5500	3.2	1.05	0.98	Sherman
16	18	13350	2	1.47	1.55	Forbes

The agreement happens to be close, much closer than can ordinarily be expected. As previously pointed out, the rate of deterioration depends much on the quality of the water carried and the above rule is offered only as a means of rough approximation.

It is found in the case of wood pipes that f is little affected by age if the pipe is clean of dirt deposits and organic growths. In some cases f has been found to decrease slightly with age.

Example.—A straight 12-inch cast-iron pipe, 4000 ft. long, is connected with a reservoir containing water whose surface level is 100 feet above the discharging end. Compute the probable velocity of flow.

From equation (96)

$$h = \frac{v^2}{2g} + 0.5\frac{v^2}{2g} + f\frac{l}{d}\frac{v^2}{2g}.$$

Inasmuch as the value of f depends upon the velocity which is unknown, we shall be obliged to first *assume* f and having found a roughly approximate value for the velocity, use this latter to obtain a closer value of f. Further trials may then be made if conditions warrant. Assuming $f = .02$,

$$100 = \frac{v^2}{64.4} + \frac{0.5\,v^2}{64.4} + \frac{0.02 \times 4000}{1}\quad\frac{v^2}{64.4}$$

$$v = 8.9 \text{ ft. per second.}$$

According to the table of the preceding article, this indicates a value for f of about .0173. Using this value, we find

$$v = 9.5 \text{ ft. per second.}$$

The corresponding value of f is .0171 which gives

$$v = 9.6 \text{ ft. per second.}$$

For all practical purposes the previous result of 9.5 ft. per second contains all the accuracy which our knowledge of the factor f warrants.

An idea of the relative size of the friction losses may be obtained from the following summary,

Velocity head =	$\frac{v^2}{2g}$	= 1.43
Entrance loss =	$\frac{.5\,v^2}{2g}$	= 0.72
Friction loss =	$f\frac{l}{d}\frac{v^2}{g}$	= 97.85
	Total	100.00

It is seen that for this particular pipe nearly 98 per cent. of the head was lost by pipe friction, while less than one per cent. was lost at entrance. Had the latter loss been neglected in our computation, no appreciable change in the computed velocity would have resulted, and it is common practice in long pipes to neglect this small loss. Just what may be considered to be a "long pipe" is rather hard to state, but in general a length of 1000 diameters should render the loss by entrance negligible.

91. Critical Velocity in Pipes.—In Art. 34 it was pointed out that water in motion along a pipe may flow in either of two widely differing ways, depending on the conditions present. As it commonly occurs, the flow may be described as turbulent or *sinuous*, the separate particles following paths that are winding and intricately interlaced. Each particle, in general, has two components to its velocity, one parallel and the other normal to the pipe's axis. These components constantly change in magnitude as the particle changes its position in the pipe's cross-section. The parallel or forward component reaches its maximum value when the particle is at or near the center of the pipe, and its minimum value when at the side of the pipe. This is due to the frictional resistance offered by the pipe's wall and to the viscosity of the water itself. Since the particles are constantly changing their position and their *forward* velocity, it will be found that their *average* forward velocity while passing through a long pipe is about the same, and that the particles to a certain extent keep company with each other in their progress along the pipe.

When in large pipes the velocity is very low, and comparatively low in small pipes, the sinuous motion is no longer found, each particle moving along the pipe in a straight line parallel to the axis. At the wall, where resistance to flow is a maximum, the velocity is zero. At the center the velocity is a maximum, with a gradual decrease taking place, as the wall is approached, due to viscosity.

The two kinds of motion descibed were demonstrated in 1883 by Professor Osborne Reynolds of England in some experiments on the flow of water through a glass tube. He showed that at low velocities the water particles passed through the tube along straight parallel paths. This motion continued under gradually increased velocities until a velocity was reached at which it was

noticed that portions of the tube remote from the entrance contained small eddyings. The presence of these eddies was noted by watching the progress through the tube of a fine stream of dye-colored water which had been injected into its entrance end. At the low velocities, the stream of dye passed through the tube presenting the appearance of a mercury column in a thermometer case. When the velocity was reached that produced the eddies just mentioned, the stream of dye wavered and broke at the point of eddying. As the velocity was further increased, the eddies filled the whole tube and the dye as a stream disappeared. The velocity at which the eddying occurred was the velocity at which straight line motion changed to sinuous. It was a *critical* velocity, and because there exists *two* critical velocities, the one just described is known as the *Higher* critical velocity. Reynolds also demonstrated that sinuous motion could be changed to straight line motion by gradually reducing the velocity through his tube. The change was always accomplished (temperature constant) at a velocity somewhat lower than that which marked the change from straight line to sinuous motion, and for this reason this second critical velocity is known as the *Lower* critical velocity.

Below the lower critical velocity, the motion is always straight line, and above the higher critical velocity, it is always sinuous. For velocities between the two, the motion is unstable and may be either straight line or sinuous according to the initial condition of flow. From his experiments, Reynolds concluded that the critical velocity is inversely proportional to the diameter of the pipe and the temperature of the water, and may be expressed as follows:

$$\text{Higher Critical Velocity} = \frac{0.246\,P}{d}. \quad . \quad . \quad (97)$$

$$\text{Lower Critical Velocity} = \frac{0.0388\,P}{d}. \quad . \quad . \quad (98)$$

the value of P being

$$P = \frac{1}{1 + 0.03368\,T + 0.000221\,T^2},$$

and d being the diameter in feet. The temperature T is measured in degrees Centigrade.

[Degrees Centigrade = (Degrees Fahr. — 32°) ÷ 1.8] With water at 10° Cent. or 50° Fahr.,

$$P = \frac{1}{1.359},$$

and

$$v_h d = 0.182$$

$$v_l d = 0.029.$$

For this temperature, motion will be

Straight line, if $vd < 0.029$,

Sinuous, if $vd > 0.182$.

For pipes of various diameters, the values of the critical velocity at this temperature are computed and shown below.

	½ in.	1 in.	2 in.	4 in.	6 in.	12 in.
Higher.....	4.37	2.18	1.09	.55	.36	.18
Lower.....	0.70	0.35	0.17	.09	.06	.03

It is generally the *lower* critical velocity that interests the engineer, as water almost invariably enters a pipe in a turbulent state. From the above table, it can be seen that pipe velocities with which the engineer has to deal are usually above the lower critical, and the flow is sinuous. It is doubtful if Reynolds' equation is reliable for pipes larger than 2 inches diameter, this being the limit of his experiments.

92. Relation Between Frictional Resistance and Velocity.— We have seen in Art. 89, equation (94), that a general expression for head lost by pipe friction is

$$h_f = K \frac{l v^n}{d^x}$$

in which K is a constant dependent for its value on the roughness of the pipe. For a pipe of given diameter and length, the equation becomes

$$h_f = k v^n. \quad . \quad . \quad . \quad . \quad . \quad . \quad (99)$$

both k and n being constants whose value may be determined as follows:

The equation may be written

$$\log h_f = \log k + n \log v$$

which is the equation of a straight line, since it is in the form of

$$y = a + nx.$$

If now we experimentally determine corresponding values of v and h_f for a wide range of velocities, and plot the logarithms of these quantities as abscissæ and ordinates respectively, we will obtain, as their locus, a straight line making some angle α with the axis of $\log v$, and cutting off an intercept on the axis of $\log h_f$ equal to $\log k$. The value of n will then be obtained from $\tan \alpha = n$.

Figure 88 shows such a plot as made by Professor Reynolds from data obtained by experiments on a small lead pipe. As the velocities were increased from very small values, Reynolds found that the plotted points, up to a certain point A, lay on a straight line which made an angle of 45° with the $\log v$ axis. The value of n for this line being unity, it will be seen that for these very low velocities, the loss in head was *directly proportional* to the *first power* of the velocity; and, since the velocities were all below the critical, it follows that the frictional resistance in straight line flow varies as the *first power* of the velocity. At A, the position of the points indicated a more rapid increase in loss with increase in velocity, but no definite law governing the change. Not until a point B was reached and passed did the law become apparent, and then it was found that further increase in v caused all the points to lie on a straight line BC. This line had such a slope that $\tan \alpha$ equaled 1.722, showing that h_f varied as $v^{1.722}$. The point A marks the *higher* critical velocity at which the *straight line* motion,

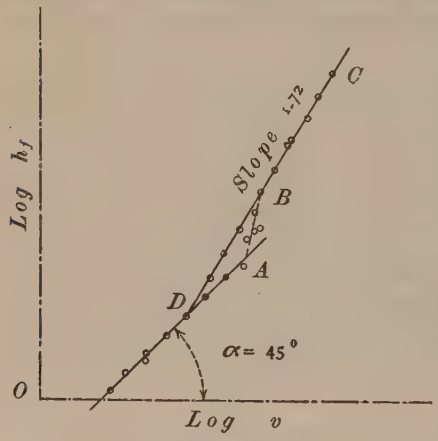


FIG. 88

which the water previously had, changed to *sinuous* motion. Reynolds found also that by starting with high velocities and gradually decreasing them, the plotted points followed the line BC down to a point D, the intersection of BC and AD. This point marks the *lower* critical velocity, at which motion that had previously been *sinuous*, became *straight line*.

By similar experiments and methods the value of n has been obtained for a great many pipes differing in material and roughness of surface. All the experiments indicate that $n = 1$, when the flow is straight line. For pipes of like material and of the same degree of roughness, n is a constant. Usually any given material varies somewhat in roughness and n then differs in value. Thus Barnes found from a study of many experiments upon lead pipe that a mean value of 1.692 was more representative of facts than Reynolds' value of 1.722.

Since the intercept of the line DC on the axis of log h_f is log k, the value of k is easily obtainable. From the equations at the first of this article,

$$k = \frac{Kl}{d^x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (100)$$

This equation contains two unknowns, K and x. The value for these may be found by experimenting with a lead pipe of the same quality of smoothness, but of *different diameter*, and obtaining a new value of k from logarithmic plotting of results. Two pair of simultaneous values for k and d being now available, two simultaneous equations may be written with the aid of (100) and the value of K and x determined.

For lead pipes, Barnes obtained mean values for K and x of 0.000486 and 1.149 respectively, giving

$$h_f = \frac{.000486\, l\, v^{1.692}}{d^{1.149}}$$

Although this expression is very much more cumbersome than the Chezy formula,

$$h_f = f \frac{l}{d} \frac{v^2}{2g},$$

it is much more representative of facts and, when the values of the constants have been verified by further experiments, it should

be more reliable. Such formulas, known as *exponential formulas*, have been determined for pipes of other material and will be discussed later. The simplicity of the Chezy formula will cause it to be ever popular and there is no valid reason against its use when satisfactory tables of f can be established. Once a satisfactory exponential formula for any particular kind of pipe has been determined, it may be equated to the Chezy formula and a solution for f made. Thus for the case cited above, we may write

$$\frac{.000486\, l\, v^{1.692}}{d^{1.149}} = f \frac{l}{d} \frac{v^2}{2g}$$

or

$$f = \frac{.0313}{v^{0.308}\, d^{0.149}}$$

Then for various values of v and d, corresponding values for f may be computed and tables constructed. This method was followed in preparing the tables in Art. 90.

93. Other Losses in Head.—Remembering that any sudden change in velocity, either in magnitude or direction, will produce a loss in head, we see that abrupt changes in diameter, and the presence of curves or elbows, gates and valves are further sources of loss. Detailed treatment of the head lost in each of these cases follows.

94. Loss by Sudden Enlargement.—If the cross-section of the pipe be abruptly enlarged, as in Fig. 89, the velocity will be suddenly reduced from v_1 to v_2 and a loss in head will result from the impact and eddying caused by the meeting of the more swiftly moving water in the small pipe with the slower water in the large pipe. A rational expression for the magnitude of the loss may be derived as follows.

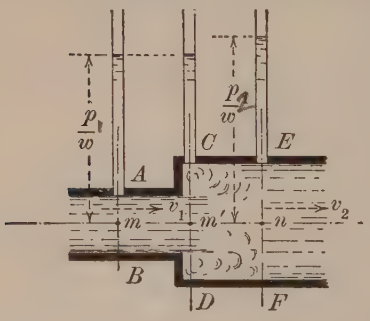


FIG. 89

Let the area of the pipe sections be a_1 and a_2, respectively, so that $a_1 v_1 = a_2 v_2$. Between the points m and n we may write

$$\frac{v_1^2}{2g} + \frac{p_1}{w} = \frac{v_2^2}{2g} + \frac{p_2}{w} + \text{Lost Head}$$

from which

$$\text{Lost Head} = \frac{v_1^2}{2g} - \frac{v_2^2}{2g} - \left(\frac{p_2}{w} - \frac{p_1}{w}\right). \quad . \quad . \quad (101)$$

The velocity v_1 is maintained up to, and probably slightly beyond *C-D* so that at m' the average pressure may be *assumed* to be p_1. The total pressure in the large pipe on section *C-D* may then be taken as a_2p_1, while on *E-F* it will be a_2p_2. That p_2 is greater than p_1 may be seen from the fact that between m and n there occurs a large decrease in velocity head without a corresponding gain in potential head. It is true that between these same points a loss is occurring, but experiments have proved that this is never large enough to cause p_2 to be less than p_1. There exists, therefore an unbalanced, horizontal force $(a_2p_2\text{-}a_1p_1)$ between *C-D* and *E-F* against which W pounds of water move each second and thereby have their velocity changed from v_1 to v_2. Since a force is proportional to the change in momentum which it can produce in a given mass in a second's time, we may write,

$$a_2p_2 - a_2p_1 = \frac{W}{g}(v_1 - v_2)$$

or

$$\frac{p_2}{w} - \frac{p_1}{w} = \frac{v_2}{g}(v_1 - v_2). \quad . \quad . \quad . \quad . \quad . \quad (102)$$

Combining (101) and (102),

$$\text{Lost Head} = \frac{v_1^2}{2g} - \frac{v_2^2}{2g} - \frac{v_2}{g}(v_1 - v_2) = \left(\frac{v_1 - v_2}{2g}\right)^2 \quad . \quad (103)$$

Another form of expression may be derived from (103) using the relation $a_1v_1 = a_2v_2$. If we substitute for v_1 its equivalent $\frac{a_2v_2}{a_1}$, there results

$$\text{Lost Head} = \frac{\left(\frac{a_2v_2}{a_1} - v_2\right)^2}{2g} = \left(\frac{a_2}{a_1} - 1\right)^2 \frac{v_2^2}{2g}. \quad . \quad (104)$$

Equation (103) will be found more generally applicable than (104)

The difference in the pressures existing in the two sections of

the pipe on either side of the enlargement may be found by substituting the value of the loss, as given by (103), in equation (101) and solving for $\frac{p_2}{w} - \frac{p_1}{w}$. There results,

$$\frac{p_2}{w} - \frac{p_1}{w} = v_2 \frac{(v_1 - v_2)}{g} \quad . \quad . \quad . \quad . \quad . \quad (105)$$

showing that the pressure always rises in passing an enlargement.

There is no positive evidence that all the assumptions made in the above derivation have the nature of facts, but it has been found that values obtained from experiments check fairly well with those obtained from the derived formula. Empirical formulas based on experiment have been offered by several investigators, but it is doubtful that they give results more trustworthy in general than (103) or (104). Thus Archer, as a result of experiments with enlargements in smooth brass pipes, proposed

$$\text{Lost Head} = 1.098 \frac{(v_1 - v_2)^{1.919}}{2\,g} \quad . \quad . \quad . \quad . \quad (106)$$

Except for low velocities the formula gives results in general slightly smaller than (103). Archer's pipes were small, ranging in diameter from about 1 to 3 inches. The velocities used were relatively very high, varying from 3.8 to 31.6 ft. per second in the first or smaller pipe. Very few runs at velocities less than six feet per second gave reliable results according to Archer's account. It is a matter of doubt whether experiments on such small pipes with high velocities may be used to predict results to be obtained from larger pipes having normal values of velocity. In detecting and measuring the loss by enlargement, Archer followed the usual custom of placing a piezometer in the smaller pipe close to the enlargement and another in the larger pipe at some distance below the enlargement. The pressure being noted for any given rate of flow, Bernoulli's Equation at once gives the *total* loss in head, between piezometers. From this must be subtracted that amount which has been caused by pipe friction. Just how far the eddying produced by the enlargement will extend downstream is problematical, but there is enough evidence to show that it may easily in some cases still exist at distances equal to 50 or more diameters. As long as the disturbance lasts, the loss in head will continue to increase in magnitude. Archer's downstream piezometer was

placed from 14 to 17 diameters below the enlargement and, in correcting for pipe friction, use was made of the Hazen-Williams pipe-friction formula which contains an empirical constant whose value has to be estimated for any particular pipe. (See Art. 105.) It seems doubtful under the circumstances whether Archer's formula is as accurate as its refined coefficients and exponents would indicate. Equation (103) is far more convenient to use and since it gives results generally a little larger than Archer's, its use in design is attended with a feeling that the loss is over-, rather than under-estimated. Again, the loss in pipe lines of any considerable length will be found to consist largely of that caused by pipe friction, and losses by enlargement or other causes generally become insignificant unless frequent in their occurrence.

Exit Loss.—Whenever a pipe discharges into a reservoir with its end beneath the water surface, it will be seen that the kinetic energy of the water is dissipated in the eddying and impact that is produced, and as the kinetic energy per pound is $\frac{v^2}{2g}$, this is the measure of the lost head. It is really a special case of enlargement where the velocity in the second pipe is zero, or

$$\text{Lost Head} = \frac{(v - 0)^2}{2g} = \frac{v^2}{2g}$$

95. Loss by Sudden Contraction.—If a pipe suffers a sudden contraction of section in the direction of the flow, a loss results which may, to a certain extent, be mathematically analyzed. It is caused (Fig. 90) by impact and eddying in the vicinity of the vertical shoulder, AB, by friction as the water passes the sharp edge ab, and by enlargement as the stream expands from the contracted section at m to fill the entire section of the second pipe. Doubtless, the major portion of the loss is due to the expansion beyond m. Since the amount of the contraction at m depends upon the distance Aa or Bb (see Art. 50), it follows that the loss by expansion will depend upon the ratio of the two diameters, d_1 and d_2, as will the entire loss due to change of section. Treating the entire loss as due to passing an orifice edge and to subsequent sudden enlargement, we may write,

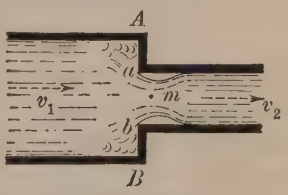


FIG. 90

$$\text{Lost Head} = \left(\frac{1}{{c_v}^2} - 1\right)\frac{{v_m}^2}{2g} + \left(\frac{a_2}{a_m} - 1\right)^2 \frac{{v_2}^2}{2g}. \quad . \quad (107)$$

The value of v_m may be expressed as $\frac{v_2 a_2}{a_m}$ or

$$v_m = \frac{v_2}{c_c}$$

c_c being the amount of contraction which results from passing the edge a-b. Using this value for v_m and noting that $\frac{a_2}{a_m} = \frac{1}{c_c}$, equation (107) may be written

$$\text{Lost Head} = \left[\frac{1}{{c_c}^2}\left(\frac{1}{{c_v}^2} - 1\right) + \left(\frac{1}{c_c} - 1\right)^2\right]\frac{{v_2}^2}{2g}. \quad . \quad (108)$$

The value of c_v would in general be about 0.98 and the value of c_c may be approximated from measurements made by several investigators. Both Freeman and Weisbach made such measurements and their results agree remarkably well. Using Freeman's values and equation (108), Hughes and Safford have computed the following values for K_c in the equation

$$\text{Lost Head} = K_c \frac{{v_2}^2}{2g}. \quad . \quad . \quad . \quad . \quad (109)$$

For $\frac{A_2}{A_1}$ =	.10	.20	.30	.40	.50	.60	.70	.80	.90	1.00
K_c =	.47	.43	.39	.34	.30	.26	.21	.16	.08	0

Certain experimental observations made by the writer lead him to believe these values to be a little large; but the scarcity of experimental data on this loss leads him to recommend the above table. As pointed out in the previous paragraph the loss is small compared to that occurring by pipe friction and its exact value in a long pipe is of secondary concern unless it occurs frequently.

96. Loss by Bends.—When water passes around a bend in a pipe, a loss in head results which occurs not only within the bend itself but also for a considerable distance in the straight pipe which follows it. Within the bend, the quick change in direction which the water makes, doubtless causes impact and a sudden rearrange-

ment of separate particles, which results in eddys that continue to dissipate energy throughout a portion of the straight pipe below the bend. Any experimental work to determine the magnitude of the loss must, therefore, include a length of this pipe sufficient to allow a return to normal flow conditions. From the total loss of head observed in the bend and the straight portion, may then be subtracted the loss occurring in the same distance by pipe friction, and the loss produced by the bend obtained. The total loss is generally measured by piezometers placed just above the

Bends in 13-foot Penstocks, Berlin, N. H. (Courtesy of Continental Pipe Mfg. Co.)

bend and at the proper distance below it. The drop in pressure between the piezometers for any given velocity will then allow the loss to be computed.

From his work on a 6-inch pipe, Professor Schoder, of Cornell, concluded that disturbances produced by a bend of 90° might continue down the pipe for a distance of 75 diameters.

Among the experiments made upon the effect of bends, the most reliable are those of Williams, Brightmore and Schoder. Williams experimented in 1893–98 with the cast-iron water mains of Detroit and measured the losses in head in bends of 90°. The sizes of pipe used were 12, 16 and 30 inches in diameter. Professor

Schoder's work was with bends of 90° in 6-inch pipes and Brightmore used 90° bends in both 3 and 4 inch pipes. A few experiments by Alexander upon varnished wood pipe $1\frac{1}{4}$ inches in diameter are also available. A study of the combined data shows it to be somewhat discordant and no satisfactory expression for relating the magnitude of the loss and the variable factors has been obtained. In 1913, W. E. Fuller published in the "Journal of the New England Waterworks Association" the results of his study of the data, including a formula for computing the loss. He proposed

$$\text{Lost Head} = kv^{2.25} \quad . \quad . \quad . \quad . \quad . \quad (110)$$

and showed by a curve the relation between k and the radius of curvature of the pipe's axis. The following simultaneous values of k and R are taken from his curve.

R in ft.	0.5	1	2	6	8	10	15	20	25	30	40	50	60
k	.0040	.0028	.0022	.0023	.0027	.0033	.0048	.0059	.0063	.0070	.0075	.0080	.0086

As it is more convenient to express the loss in terms of the velocity head, we may write

$$\text{Lost Head} = K_b \frac{v^2}{2g} \quad . \quad . \quad . \quad . \quad . \quad (111)$$

and obtain values for K_b by equating (110) and (111). The following table gives K_b for various values of v and R.

VALUES FOR K_b FOR BENDS

VEL. IN FT.	RADIUS OF BEND IN FEET												
	0.5	1	2	6	8	10	15	20	25	30	40	50	60
1	.26	.18	.14	.15	.17	.21	.31	.38	.41	.45	.48	.52	.55
2	.31	.21	.17	.18	.21	.25	.37	.45	.48	.54	.58	.61	.66
3	.34	.24	.19	.20	.23	.28	.41	.50	.53	.59	.64	.68	.73
4	.36	.26	.20	.21	.25	.30	.44	.53	.57	.64	.68	.73	.78
5	.38	.27	.21	.22	.26	.32	.46	.56	.60	.68	.72	.77	.83
6	.40	.28	.22	.23	.27	.33	.49	.59	.63	.71	.76	.81	.87
8	.43	.30	.24	.25	.29	.36	.52	.63	.68	.76	.81	.87	.93
10	.46	.32	.25	.26	.31	.38	.55	.67	.72	.80	.86	.92	.99
15	.50	.36	.28	.29	.34	.42	.61	.74	.80	.89	.95	1.02	1.09
20	.54	.38	.30	.31	.37	.45	.66	.80	.86	.95	1.02	1.09	1.17

The figures in the table bear out the conclusion, separately reached by the different experimentors, that the loss decreases with increased radius of bend down to a certain point, and then increases. The above figures indicate that the loss is a minimum when the radius of the bend is from 2 to 6 feet.

Equation (110) is based upon a curve which Fuller plotted as an *average* of the curves obtained by each investigator, and, therefore, neither the equation nor the values in the tables can be regarded as more than approximate. For velocities up to about 5 or 6 feet per second his average curve closely followed the others and values of lost head computed by this method should give good results. More experiments are needed, particularly upon large sized pipes in order to validate (110) or show in what way it should be modified.

Little or nothing is known of the relation between the losses in bends of the same radius as the angle subtended by the bend decreases. It is probable that for a 45° bend the loss is only slightly smaller than for a bend of 90°.

Loss Caused by Valves.—A valve partly closed is an obstruction to flow, causing loss of head by sudden change of section.

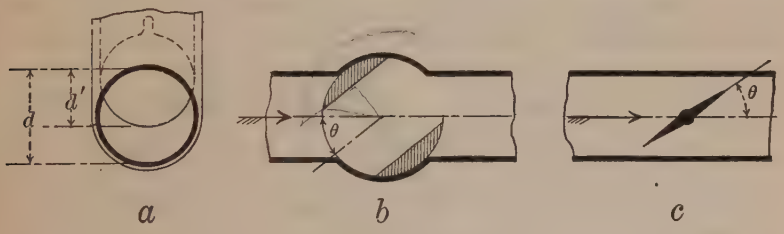


FIG. 91

This is really one of the functions of a valve as usually the only way in which the rate of flow in a given pipe can be regulated is by varying the amount of available head. When wide open, a good valve should offer little or no resistance to flow, but most valves, by reason of their design, do cause a loss in their open position. This is especially true of the valves commonly used in small pipes less than 3 inches in diameter. The data on valve loss are quite meagre and few figures are available on the loss when the valve is wide open. In the following paragraphs are given figures regarding losses in the three types of valve illustrated in Fig. 91. The formula generally adopted is

$$\text{Lost Head} = K_v \frac{v^2}{2g}. \quad . \quad . \quad . \quad . \quad . \quad (112)$$

the value of K_v being experimentally determined.

(a) *Gate Valves in Circular Pipes* (Fig. 91a).—The best information which we have concerning the loss produced by partial closure of a gate valve is supplied by data obtained by E. Kuichling in 1891 from experiments upon a 24-inch valve in the Rochester pipe line (Trans. A. S. C. E., vol. 26, 1892). The following values, compiled by the writer, indicate probable values of K_v for different ratios of $\frac{d'}{d}$.

For $\frac{d'}{d}$ =	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
K_v =	.4	.6	.7	1.0	1.8	3.7	7.6	15.6	35	100

The figures will be found to differ slightly from those appearing in several texts, due to the fact that they include *all* losses caused by the presence of the valve in the pipe, while the figures of Kuichling and others represent only the loss caused by the valve disk and do not include that produced by the seat or groove. This latter loss, while not large, should be included if the real effect of the valve is to be formulated. The detailed report by Kuichling permits the computation of this loss and the writer has included its effect in the above table. Values of K_v for ratios of $\frac{d'}{d}$ not observed by Kuichling were obtained by direct plotting and curve exterpolation.

(b) *Cocks in Circular Pipes.*—(Fig. 91b.) For a cock valve Weisbach found the following values of K_v for various angles, θ, of closure.

For θ =	5°	10°	15°	20°	25°	30°	35°	40°	45°
K_v =	0.05	0.3	0.75	1.56	3.1	5.47	9.68	17.3	31.2

θ =	50	55°	60°	65°	82°
K_v =	52.6	106	206	486	Valve closed

(c) *Butterfly Valve.*—(Fig. 91c). These figures are from Weisbach's work on pipes and valves less than 2 inches in diameter.

For θ =	5°	10°	20°	30°	40°	50°	60°	70°
K_v =	0.24	0.52	1.54	3.91	10.8	32.6	118	751

98. Loss Caused by Pipe Fittings.—The presence of elbows, tees, Y's and other common pipe fittings add to the head consumed in friction. Accounts of experiments upon standard fittings have from time to time appeared in engineering publications, but a résumé of them can hardly be given in an elementary text of the present character. Gibson in his "Hydraulics," pp. 256–257, gives the results of several investigations on such fittings. For elbows in pipes having diameters from 3 to 6 inches, the tables given indicate a loss approximately equal to $0.6\,\frac{v^2}{2g}$; while in pipes 1 to 2 inches in diameter, the loss varies from 1 to 1.35 times $\frac{v^2}{2g}$.

99. Long Pipe with Changes in Diameter, Bends and Valves.—The application of the foregoing principles may be seen in the

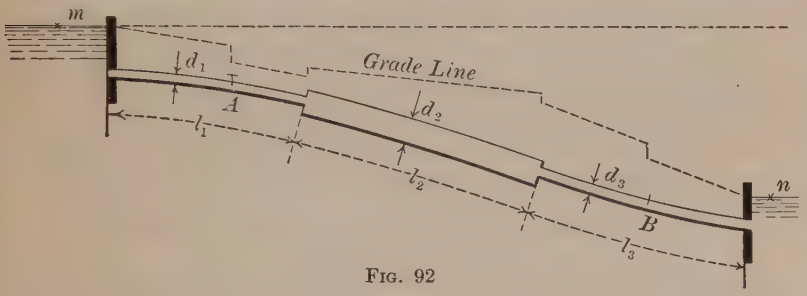


FIG. 92

case of the pipe line illustrated in Fig. 92 where two reservoirs are connected by pipe which is made up of three sizes, d_1, d_2 and d_3 ft. in diameter, the change of section being abrupt in each case. The first length is fitted with a gate valve at A and the third length contains two 90 degree bends at B made with a radius of 10 feet. With the valve half closed $\left(\frac{d'}{d} = 0.50\right)$ it is desired to ascertain the rate of flow into the second reservoir.

If we apply Bernoulli's equation between the points m and n, we see that the entire head h is lost in friction since velocity head and pressure-head at both points are equal to zero.

We may, therefore, write,

$$h = f_1 \frac{l_1}{d_1}\frac{v_1^2}{2g} + f_2 \frac{l_2}{d_2}\frac{v_2^2}{2g} + f_3 \frac{l_3}{d_3}\frac{v_3^2}{2g} + \frac{.5\,v_1^2}{2g} + K_v \frac{v_1^2}{2g} + \frac{(v_1 - v_2)^2}{2g} + K_c \frac{v_3^2}{2g} + 2\,K_b \frac{v_3^2}{2g} + \frac{v_3^2}{2g}. \quad . \quad . \quad . \quad (113)$$

the losses, in their order of enumeration, being due to pipe friction, entrance, passing valve, enlargement, sudden contraction, bends and exit or discharge loss. Had the point n been taken at the mouth of the last pipe, the equation obtained would have been identical with the above. The potential head, h, at the point m would have been increased by e, the distance that the end of the pipe is submerged below the level of the second reservoir. The same distance, e, would have represented the pressure-head at n and cancellation of e from both sides of the equation would have followed. The velocity head at n would have been $\frac{{v_3}^2}{2g}$ and, as the discharge loss takes place *beyond* n, the last term of the above equation would have represented the velocity head at n. (The student should write the equation.) We now see that the amount of submergence of the pipe's end has no effect upon the discharge, the latter depending solely on the difference between reservoir levels and the head lost. Similarly the discharge is not affected by the amount of submergence of the first pipe below the level of the upper reservoir, provided there is enough head over the entrance to give the water the necessary initial velocity as it enters the pipe. The minimum head must equal $1.5\ \frac{{v_1}^2}{2g}$ plus whatever pressure-head it is desired to maintain in the pipe just beyond the entrance point. (See next Article). With large pipes a greater submergence than this is necessary in order to prevent air being carried into the pipe by means of vortices which would otherwise be formed in the water surface immediately over the entrance.

For the case of the pipe in Fig. 92, the following dimensions will be assumed:

$h = 100$ ft., $d_1 = 6$ in., $d_2 = 12$ in., $d_3 = 8$ in., $l = 1000$ ft., $l_2 = 3000$ ft., $l_3 = 2000$ ft.

Since the velocities are unknown we have no exact knowledge of the pipe-friction factors or of the quantities K_b and K_c. The first may be tentatively assumed as 0.02 in value, and as it is probable that the loss by pipe friction is a very large percentage of the whole loss, we may save labor by temporarily neglecting the others and determining v_1, v_2 and v_3 approximately. In this way close values for the friction factors may be quickly obtained.

We will, therefore, write

$$100 = \frac{.02 \times 1000}{\frac{1}{2}} \cdot \frac{v_1^2}{2g} + \frac{.02 \times 3000}{1} \cdot \frac{v_2^2}{2g} + \frac{.02 \times 2000}{\frac{3}{8}} \cdot \frac{v_3^2}{2g}$$

and replace v_2 and v_3 by $\frac{v_1}{4}$ and $\frac{9}{16} v_1$, which we obtain from the equation, $a_1 v_1 = a_2 v_2 = a_3 v_3$. There results,

$$\frac{v_1^2}{2g} = 1.6 \text{ ft. or } v_1 = 10.2 \text{ ft. per sec.}$$
$$v_2 = 2.6 \text{ ft. per sec.}$$
$$v_3 = 5.8 \text{ ft. per sec.}$$

Referring to the table of friction factors on page 157, the corresponding values of f are,—

$$f = 0.019$$
$$f = 0.0207$$
$$f = 0.0196$$

For K_b and K_c, the values indicated by the above velocities are 0.33 and 0.32 respectively, while K_v has the value 3.7. If these values of K and f be used in equation (113), there results,

$$v_1 = 9.92$$
$$v_2 = 2.48$$
$$v_3 = 5.58$$

for which values the f's are practically the same as before and no further substitution is necessary. The values of the losses are as follows:

Friction loss (1st pipe)	=	57.70 ft.
" " (2nd pipe)	=	5.90 ft.
" " (3rd pipe)	=	28.24 ft.
Entrance loss	=	0.76 ft.
Valve loss	=	5.62 ft.
Enlargement loss	=	0.86 ft.
Contraction loss	=	0.15 ft.
2 Bend losses	=	0.32 ft.
Discharge loss	=	0.45 ft.
Total	=	100.00 ft.

It is seen that the loss by pipe friction and the valve is nearly 97.5 per cent. of the whole. In fact, had the other losses been neglected the velocities would have figured as 10.0 ft., 2.5 ft., and 5.62 ft. per second. It is generally true that in a very long pipe losses by changes of section and bends may be neglected if

not of frequent occurrence. Similarly the loss at entrance and the discharge loss (or velocity head if the pipe discharges freely into the air) are commonly neglected for such pipes, and the available head h equated to the loss by pipe friction giving

$$h = f\frac{l}{d}\frac{v^2}{2g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (114)$$

Of course this relation cannot be true but the error involved is usually smaller than that arising from a choice of the value for f, and its simplicity has caused it to be much used as a formula for pipe flow. (See Art. 104.)

100. Hydraulic Gradient.—The Hydraulic Gradient, or Grade Line, may be defined as an imaginary line drawn above a pipe

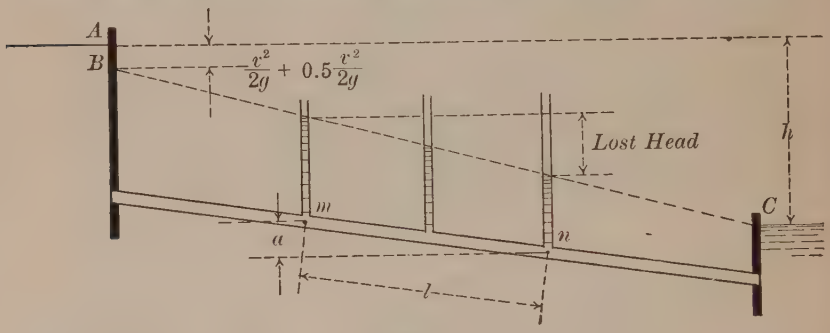


Fig. 93

(and in the same vertical plane as its axis) so that the vertical distance from any point on the pipe's axis to the line represents the pressure-head at that point. In other words, if a row of open piezometers were placed at frequent intervals along a pipe line, the grade line would join the levels of their water columns. For a *straight* pipe of *uniform section*, the grade line will be a straight line, sloping in the direction of flow as may be seen from a study of the pipe shown in Fig. 93. Two piezometer columns at any points, m and n, indicate the pressure heads at the respective points. The velocity being constant,

$$\frac{p_m}{w} + a = \frac{p_n}{w} + \text{Lost Head between points,}$$

and

$$\text{Lost Head} = \left(\frac{p_m}{w} + a\right) - \frac{p_n}{w},$$

or is equal to the difference in level of the tops of the two piezometer columns. Since the length of the pipe separating the two columns is the only factor affecting the magnitude of the loss, it is apparent that the vertical distance between the tops of these columns is directly proportional to the intervening length. The line joining the tops is therefore straight.

At the discharge end of the pipe, the pressure-head is equal to the depth of submergence below the reservoir level, and the grade line terminates at C. Had the pipe discharged into the air, the pressure at the end would have been zero, and the grade line would have run to the end of the pipe.

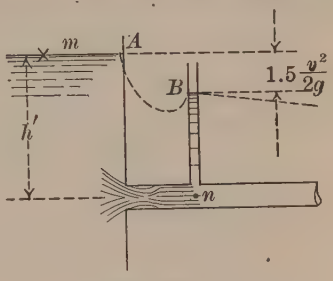


FIG. 94

Close to the upper reservoir, the gradient ceases to be straight owing to the rapid change in pressure-head which takes place in the first few diameters of the pipe's length (Fig. 94). Written between m and n, Bernoulli's Equation gives

$$h' = \frac{v^2}{2g} + \frac{p}{w} + \text{Entrance Loss.}$$

or

$$\frac{p}{w} = h' - \left(\frac{v^2}{2g} + \text{Entrance Loss}\right).$$

The point B is therefore below A by the amount of the velocity head at n plus the head lost in entering the pipe. Between A and B a curved line may serve to represent the gradient. In long pipes where the greater portion of the available head is lost in friction, the vertical distance, AB, is relatively so small that the curved portion of the line is usually neglected and the gradient assumed to run straight from A.

Pipe lines, which are horizontal or have a straight profile as shown in Fig. 93, may be so constructed as to contain changes in a horizontal direction. Since the grade line is always above the pipe, it could not under these conditions be straight, but would have a *uniform rate of slope* if the pipe were constant in section. Pipes which are fairly straight in plan may contain slight changes in vertical elevation without sensibly changing the gradient from a straight line, provided the changes are made by gradual curves

and the length of the pipe be practically the same as that of the straight line joining its ends.

Sharp curves or bends will depress the grade line owing to the loss in head produced and, generally speaking, the grade line will be depressed wherever a loss occurs. An exception to this rule is found in the case of sudden enlargement in section (Art. 94). Reference to equation (105) will show that the pressure-head immediately following an enlargement is always greater than that just before, and consequently the grade line rises slightly.

From all the above statements the following facts regarding the grade line may be deduced:

(a) Shows, by its distance above the pipe, the pressure-head at any point in the pipe.

(b) Has a general slope in the direction of flow.

(c) In the same pipe, or pipes of like material and diameter, the steeper the slope of the grade line, the higher is the velocity of flow.

(d) Shows by its vertical drop the amount of head lost in any manner between two points in the pipe, *provided* the velocity at the two points is the same.

(e) Is practically straight for straight pipes of uniform section and practically so for pipes whose length is sensibly the same as that of the straight line joining their ends.

In Fig. 92 is sketched the grade line for a pipe containing changes in diameter, a partly closed valve at A, and bends at the point B.

101. Pipes which Run Above the Gradient.—In Fig. 95 is shown a pipe which has a portion of its length above the grade

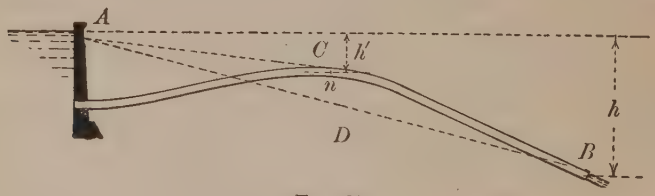


FIG. 95

line. The latter, while not quite straight, may in this discussion be considered so and be represented by the line ADB. The point *n*, lying above the gradient, has its ordinate to the gradient measured downward, indicating a pressure at this point *less than at-*

mospheric. At a point to the left, where the gradient crosses the pipe, the pressure is that of the atmosphere. If the pipe be air-tight throughout the portion above the gradient, and the velocity of flow sufficiently high, no disturbance in the rate of flow will occur and the discharge from the pipe will take place under the head h. Water generally contains a quantity of air in solution which is discharged whenever the pressure falls below that of the atmosphere. For this reason the above stipulation was made that the velocity of flow be sufficiently high in order that any air set free in the portion of the pipe above the gradient be swept along by the friction of the water and removed from the pipe. If the velocity is not sufficient, the air will gradually collect in the summit near n, and tend to raise the pressure there. This would cause the point D to be raised and the portion AD of the gradient would then have a smaller slope. From our study in the preceding article, we see that this would be accompanied by a *decrease* in flow in the corresponding portion of the pipe and therefore in the entire pipe. Under these conditions the slope of the pipe from C to B might be so great as to cause it to be only partially filled. Should the pressure at n rise to atmospheric, the gradient will shift to A-C and the discharge will take place under the reduced head h'. Under this condition the pipe from C to B could not flow full. The importance of knowing the relative positions of the gradient and pipe line is at once seen. No long pipe line having variations in its level should be laid until a gradient is plotted on the profile of its course and the absence of "summits" above the gradient assured. If the total fall of the pipe be small relative to its length, the profile of the pipe should be plotted to an exaggerated scale to allow risings above the grade line to be more readily detected. The city of Cambridge, Massachusetts, in constructing its water supply some years ago, inadvertently allowed a long pipe line to run above the gradient. The ground traversed by the pipe was rolling in character and the total fall from reservoir to reservoir, small. A profile had been prepared and seemed to show the pipe to be below the gradient, although at one point the two closely approached each other. After construction the pipe failed to deliver the desired quantity and a new profile made on an exaggerated vertical scale showed the pipe at one point lying several feet above the gradient. A relocation of this portion resulted in a substantial increase in flow.

Where the presence of joints permits air to enter summits above the grade line, increased difficulties will be met since the velocity is seldom high enough to sweep the air from the pipe. Long pipes must sometimes of necessity be laid with portions above the gradient and then the air is removed from the summits by attaching pumps which automatically keep the pressure down to the desired point. Summits may also occur at points below the grade line and air which collects may then be forced out by the water pressure through vents or valves.

Assuming an air-tight pipe, the limit to the distance which a pipe may be carried above the gradient is $\frac{p_a}{w} - \frac{p_v}{w}$, p_a being the pressure of the atmosphere and p_v the pressure of the water vapor which forms even at relatively low temperatures. From the table given in Article ~~10~~ 11, values of $\frac{p_v}{w}$ for various temperatures may be obtained. For water at 60° Fahr., the value of $\frac{p_v}{w}$ is approximately 0.6 ft., and with an atmospheric pressure of 14.7 lb. per sq. inch, the value of $\frac{p_a}{w} - \frac{p_v}{w}$ becomes 33.8 — 0.6, or 33.2 feet. At the given water temperature, a summit carried more than this distance above the gradient would cause *steady* flow to cease even though the summit were at a lower level than the supplying reservoir. This is because steady flow would demand that the pressure be less than absolute zero, which is impossible.

Siphon is a term used to designate a pipe, or a portion of a pipe, which runs above a gradient. The term *Inverted Siphon* is often applied to that portion of a pipe which crosses a valley or depression, although the pipe at this point is below the grade line and under pressure. Evidently the term is a misnomer.

102. Pipe Lines Supplied by Pump.—Pipe lines are more often furnished with water from a pump than from a gravity reservoir. The point of discharge is generally at a higher elevation than the pump and the latter supplies the head necessary to maintain the discharge. No new principle is involved in the theory of flow through the pipe under these conditions, but the location of the hydraulic gradient and the amount of head and energy furnished by the pump should be noted. A simple case is illustrated

in Fig. 96, water being pumped from a reservoir at A to a higher-level reservoir at C. When at A, the water possesses no energy or head with reference to a datum through A. As it passes through

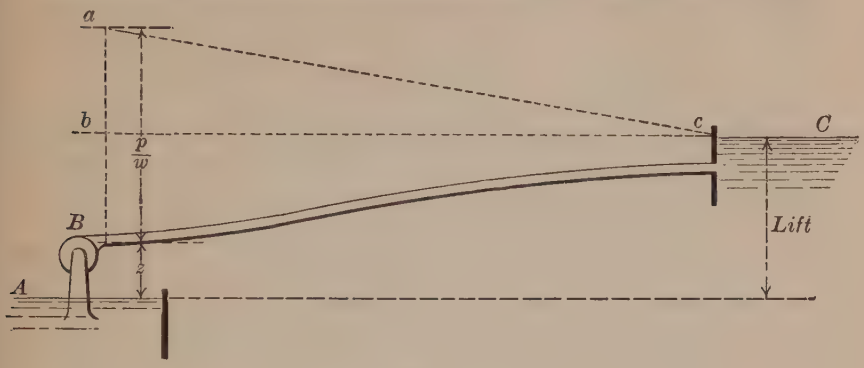


FIG. 96

the pump, work is done upon it and energy stored in it. On leaving the pump at B its energy per lb. (or head) is

$$\text{Energy per lb.} = \frac{v^2}{2g} + \frac{p}{w} + z$$

and if Qw pounds of water have passed through the pump each second, then

$$\text{Energy delivered by pump each second} = Qw\left(\frac{v^2}{2g} + \frac{p}{w} + z\right) \quad (115)$$

We see that the pump has raised the water through a distance z, given it velocity (hence kinetic energy) and raised its pressure.

If we consider the heads at the points B and C we find

$$\frac{v^2}{2g} + \frac{p}{w} + z = \text{Lift} + f\frac{l}{d}\frac{v^2}{2g} + \frac{v^2}{2g}$$

in which the last term represents the head lost by discharge into the reservoir. By reason of the equality expressed in the equation, the right-hand member represents the head (or energy per pound) received from the pump. Again we may write,

$$\text{Energy delivered by pump per second} = Qw\left(\text{Lift} + f\frac{l}{d}\frac{v^2}{2g} + \frac{v^2}{2g}\right) \quad (116)$$

and we see that the pump has raised the water from A to C, given it velocity-head and supplied the head lost in pipe friction. These are, in general, the items of work which must be credited to any pump which discharges through a pipe line. From equation (115) it is seen that a direct computation of work done can be made if the rate of discharge be known and the pressure and potential head at B be measured.

The grade line during pumping is shown as *ac*. It will be sensibly a straight line whose total fall, *ab*, equals the head lost in pipe friction (Art. 100*d*). The pressure-head at B is just sufficient to raise the water to the level of C and supply the head lost by pipe friction. The velocity head at B is lost at discharge.

Example (1). In Figure 96, let it be assumed that 7.85 cu. ft. per sec. are being pumped through 1000 ft. of 12-inch pipe to a reservoir whose level is 100 ft. above that of the supplying reservoir. From equation (116),

Energy per second

$$= (7.85 \times 62.5)\left(100 + \frac{.017 \times 1000}{1} \times \frac{100}{64.4} + \frac{100}{64.4}\right)$$

$$= 491(100 + 26.4 + 1.6)$$

$$= 62800 \text{ ft. lb.}$$

Horse power delivered $= 62800 \div 550 = 114$

If point B be 10 ft. above A,

$$\text{Pressure-head at } B = 100 + 26.4 - 10 = 116.4 \text{ ft.}$$

Example (2). With the pipe and elevations given in the preceding example, how much water can be delivered if 100 H. P. be delivered to the pump and the efficiency of the latter be 75 per cent.?

Energy per second delivered by pump $= 100 \times 550 \times .75 =$ 41250 ft. lb.

$$Q = av = 0.785\, v.$$

From equation (116)

$$41250 = (0.785\, v \times 62.5)\left(100 + \frac{.02 \times 1000}{1} \frac{v^2}{64.4} + \frac{v^2}{64.4}\right)$$

or

$$v = 7.2 \text{ ft. per second.}$$

From the table on page 157, this corresponds to a value for f of 0.0177. If a value of 0.0175 be used and the above equation resolved, we find

$$v = 7.1 \text{ ft. per second,}$$

giving a discharge of 5.6 cu. ft. per second.

103. Other Formulas for Flow in Pipes.—In a pipe whose length, as measured in diameters, is very long the loss by pipe friction constitutes so large a portion of the total head, h, that we may write (see equation 94)

$$h = \frac{Kl v^n}{d^x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (117)$$

without introducing an error larger than is generally involved in the selected values for K, n and x. The expression relates v with all the factors which influence its value and hence may be considered an equation for flow in long pipes of constant diameter. By simple transposition,

$$v = \left(\frac{1}{K}\right)^{\frac{1}{n}} d^{\frac{x}{n}} \left(\frac{h}{l}\right)^{\frac{1}{n}}$$

or

$$v = K' d^{\alpha} \left(\frac{h}{l}\right)^{\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (118),$$

the quantity $\frac{h}{l}$ representing the head lost per foot of pipe. A graphical conception of this quantity may be obtained from Fig. 93. Entrance loss and velocity head having been neglected in writing equation (117), the grade line may be assumed to start at the point A. Also we may assume the length of AC to be the same as l, the length of the pipe, this being practically true for long pipes. The slope of AC will then be expressed by $\frac{h}{l}$ since the latter measures the sine of the angle which AC makes with the horizontal. For the sake of simplicity we may replace $\frac{h}{l}$ by S and refer to it as the slope of the gradient or the *hydraulic slope* of the pipe. For a long, straight pipe laid horizontally, S very closely equals the sine of the angle of the gradient's inclination.

For other pipes this definition contains some approximation, but the approximation is one of definition only and S, or $\frac{h}{l}$, is always the loss in head per foot of length.

Equation (118) is convenient in form for use with pipes but it will later be shown that it may be applied to conduits flowing partly full or to open channels, if d be replaced by R, the *hydraulic radius* of the channel. The definition of this term, as applied to a pipe is the ratio of the pipe's cross-sectional area to its inside wetted perimeter, or for a circular pipe,

$$R = \frac{\text{Area}}{\text{Perimeter}} = \frac{\frac{\pi d^2}{4}}{\pi d} = \frac{d}{4}.$$

Since R is directly proportional to d, it may be substituted for the latter in (118) and, replacing $\frac{h}{l}$ by S, we obtain

$$v = C R^{\alpha} S^{\beta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (119)$$

The use of R in place of d allows the equation to be applied to pipes having shapes other than circular. For any particular pipe, or for pipes of like material having the same smoothness, the values of C, α and β will be found constant. If the formula be applied to pipes in general and α and β be given constant values for all kinds of pipe, C will be found to vary not only with the roughness of the pipe but also with R and S since these quantities have exponents which are only approximate in value. These last statements are important and should be mastered by the student.

Practically all formulas for pipe or channel flow have, or may be expressed in, this form. A brief review of the most important ones follows.

104. Chezy's Formula.—As early as 1775, Chezy proposed

$$v = C\sqrt{RS}. \quad . \quad . \quad . \quad . \quad . \quad (120)$$

which gives to α and β the value of 0.50.
The formula is identical with

$$h_f = f \frac{l}{d} \frac{v^2}{2g} \qquad \text{(See equation 95)},$$

which was the result of studies of Darcy and Weisbach. Solving the latter for v, replacing d by $4R$ and $\frac{h_f}{l}$ by S, we obtain

$$v = \sqrt{\frac{8g}{f}}\sqrt{RS}.$$

$$\text{Evidently } C = 2\sqrt{\frac{2g}{f}} \quad . \quad . \quad . \quad . \quad (121)$$

The formula has always received wide recognition and has been more generally used than any other. For many years it was practically the only one available and, now that others more closely expressing the law of flow have been brought forward, it is being continued in use by reason of custom and the fact that the one-half power of R and S makes its solution easy. C will be found to vary with R and S, since these quantities have not correct exponents, and also with the roughness of the pipe's interior surface. If values of C for a given kind of pipe are definitely known for a wide range of values of R and S, nothing of importance can be urged against its use, save that it is incorrect in its exponents and necessitates numerous tables of C to cover the many kinds of pipe in use. Values of C for pipes of smooth cast-iron, wrought-iron and wood, given in the following tables, are based on the values of f appearing in Art. 90.

VALUES OF C FOR CLEAN, SMOOTH, CAST-IRON PIPE

DIAM. IN INCHES	VELOCITY IN FEET									
	1	2	3	4	5	6	8	10	15	20
4	95	101	104	106	107	108	111	114	116	120
5	97	103	106	107	110	111	114	115	118	121
6	99	104	107	110	111	114	115	116	121	123
8	101	106	110	112	114	115	118	120	123	127
10	103	108	112	114	116	118	120	121	125	129
12	105	110	114	116	118	120	121	123	127	131
15	106	112	115	118	120	121	123	125	129	134
18	108	114	116	120	121	123	125	127	131	136
20	110	115	118	121	123	125	127	129	134	136
24	111	116	120	123	125	127	129	131	136	138
30	114	118	121	125	127	129	131	134	138	141
36	115	120	123	127	129	131	134	136	141	144
42	116	121	125	129	131	134	136	138	141	144
48	118	123	127	129	131	134	136	138	144	147
60	120	125	129	131	134	136	138	141	147	150
72	121	127	131	134	136	138	141	144	147	150
84	123	129	134	136	138	141	144	147	150	153
96	125	131	136	138	141	144	147	147	153	157

Values of C for Small Wrought-Iron Pipes

Nominal Diam. in Inches	Actual Diam. in Inches	Velocity in Feet									
		1	2	3	4	5	6	8	10	15	20
¾	0.824	77	81	84	86	87	88	90	92	94	96
1	1.048	79	84	86	88	89	90	92	94	97	99
1¼	1.380	81	85	88	90	91	93	94	96	99	101
1½	1.61	82	87	89	90	93	94	96	97	101	103
2	2.0	84	88	90	93	94	95	98	99	103	105
2½	2.5	85	89	92	94	96	97	100	101	104	106
3	3.0	87	90	94	96	98	99	101	103	106	108

Values of C for Wood Pipe

(a) Very Smooth,—planed. (b) Smooth,—Unplaned

Diam. in Inches		Velocity in Feet									
		1	2	3	4	5	6	8	10	15	20
12	*a*	101	112	119	124	128	131	137	142	151	158
	b	84	93	99	104	106	110	114	118	125	131
24	*a*	106	117	125	130	135	137	144	149	158	165
	b	88	97	104	108	111	114	119	123	131	137
36	*a*	108	120	127	134	137	141	147	152	161	170
	b	90	100	106	111	114	117	122	126	134	140
48	*a*	111	123	130	136	139	144	150	154	165	172
	b	92	102	108	112	116	120	125	129	137	143
60	*a*	112	124	131	137	142	146	152	157	168	176
	b	93	103	110	114	118	121	126	130	138	144
72	*a*	114	125	134	138	144	148	153	160	170	177
	b	94	104	111	115	120	122	128	132	141	147
84	*a*	115	127	135	141	145	150	155	161	172	179
	b	95	106	112	116	120	124	129	134	142	149
96	*a*	115	127	135	141	146	150	157	162	172	180
	b	96	106	112	117	121	125	130	135	143	150
108	*a*	116	128	136	142	147	151	158	163	174	182
	b	97	107	114	118	122	125	131	136	144	150
120	*a*	116	129	137	144	148	152	159	164	176	183
	b	97	107	114	120	123	126	132	136	145	151
132	*a*	118	130	138	144	150	153	161	165	176	184
	b	98	108	115	120	124	127	134	137	146	152
144	*a*	118	131	138	144	150	154	161	166	177	185
	b	99	109	115	120	125	128	134	138	147	153

105. Williams and Hazen's Formula.—In 1905 (and again in 1920) Messrs. Williams and Hazen published the results of a careful study by them of all available trustworthy experiments upon pipe flow. The pipes covered by the experiments ranged in diameter from 1 inch to about 15 ft. and the materials of construction included tin, lead, brass, wrought-iron, cast-iron, riveted steel, wood, cement, brick and glass. They proposed

$$v = C\,R^{.63}S^{.54}0.001^{-0.04}. \quad . \quad . \quad . \quad . \quad (122)$$

as a formula having exponential values for R and S more closely representing actual facts than those of Chezy. Assuming this to be so, it follows that C in their formula is more constant in value than in Chezy's and should vary mainly with the roughness of the pipe's interior. To a great extent this is probably true and the following statement regarding its value is quoted from the authors' book.

"In a general way it may be said that for cast-iron pipe, very straight and smooth, C may be as high as 140, but for ordinary conditions 130 is a fair value for new pipe. As pipes rust and become dirty, the value of C decreases, as has been mentioned above. For new riveted steel pipe C is about 110.

"In making estimates for pipe-lines where the carrying capacity after a series of years, rather than the value of the new pipe, is the controlling factor, a considerably lower value of C must be used, depending upon the amount of deterioration which is contemplated. A fair value for general computation is $C = 100$ for cast-iron pipe and $C = 95$ for steel pipe, but for small iron pipes a somewhat lower value of C should be taken. Lead, brass, tin, and glass, and other pipe presenting perfectly smooth surfaces, and perfectly straight, will give values of C up to 140. A very little falling off in the smoothness will reduce the value of C to 130 and 120, or even less. For smooth wooden pipe or wooden-stave pipe 120 seems a fair value. For masonry conduits of concrete or plastered, with very smooth surfaces, when clean, values of $C = 140$ may be observed. Generally such surfaces become slime-covered, reducing the value of C to 130 or less in a moderate length of time; and if the surfaces are only a little less smooth, say in such shape as is represented by ordinary good work, the value of C is reduced to 120. A conservative value for general use with first-class masonry structures is about 120. For brick sewers

much lower values may be used, and $C = 100$ seems safe. For vitrified pipe $C = 110$ may be used. It must be understood that these values depend entirely upon the smoothness and regularity of the surfaces, and are likely to vary in individual cases."

For a more detailed statement concerning the value of C and its probable decrease as the pipe ages, the student is referred to Williams and Hazen's "Hydraulics Tables," Wiley and Sons, New York.

Since $0.001^{-0.04}$ is equal to 1.318, the formula may be written

$$v = 1.318\, C\, R^{.63} S^{.54}$$

The constant $0.001^{-0.04}$, or 1.318, was introduced in order to equalize the value of C with the value in the Chezy formula when the value of R is unity and S equals 0.001. In other words, the value of C in the Williams-Hazen formula is the same as would be used in the Chezy formula were the value of R unity and S equal to 0.001. This value of S was chosen because hydraulic slope is commonly expressed in feet per thousand.

While comparatively new, the formula has been widely accepted. The apparent disadvantage of the odd exponents is more than offset by the added certainty which attends the selection of C. Its use is greatly facilitated by a slide rule devised by the authors.

106. Barnes' Formulas.—Since for any particular pipe, or for pipes of like material having the same smoothness, the values of C, α and β are practically constant (Art. 101), it has been proposed by A. A. Barnes of England that a separate formula for each kind of pipe be constructed. In his book, "Hydraulic Flow Reviewed," Spon and Chamberlain, N. Y., 1916, he gives a detailed account of his studies in this direction and a list of values for C, α and β for a large number of different pipes. From the latter, the following values are taken and, for convenience in computation, corresponding values of K, n and x are included, for use in the formula

$$h_f = K\, l\, \frac{v^n}{d^x}.$$

COEFFICIENTS AND EXPONENTS FOR USE IN THE BARNES FORMULA

MATERIAL OF CONSTRUCTION	C	α	β	K	n	x
New Asphalted Cast Iron	174.1	.769	.529	.000436	1.891	1.454
New Uncoated Cast Iron	136.6	.600	.512	.000343	1.953	1.172
New Asphalted Single-riveted Steel	171.4	.723	.527	.000386	1.898	1.372
" " Double-riveted Steel	129.9	.440	.520	.000279	1.923	0.846
New Lead	232.8	.679	.591	.000486	1.692	1.149
New Planed Wood or Wood-stave	223.3	.660	.586	.000098	1.707	1.126
New Unplaned Wood or Wood-stave	182.5	.666	.569	.000107	1.757	1.171
Clean Well-pointed Hard Brick	92.1	.602	.466	.000061	2.146	1.292
Smooth Concrete	95.1	.567	.471	.000063	2.123	1.204

The formulas obtained by using these values agree remarkably well with the results of the experiments from which the constants were derived. How well they will serve to give the flow through any pipe under consideration will largely depend on the similarity of the pipe to those used in the original experiments. In certain cases the experimental pipes were so diversified in diameter, construction date and locality, that the formula obtained should be wide and dependable in its application. This was notably so for asphalted cast-iron pipes and asphalted double-riveted pipes. The size of the former varied from 3 to $61\frac{1}{8}$ inches in diameter, the range of velocities was from 1 to 8.70 feet per second, and the hydraulic slope had values from 1 in 14,500 to 1 in 14.7. Of the 79 individual experiments studied and used, 36 agreed with the formula to within 1 per cent.; 60 agreed to within 2 per cent.; 72 agreed to within 3 per cent.; 76 agreed to within 4 per cent., and *all* agreed to within 4.5 per cent.

For the other pipes quoted above, the diversification was not sufficiently wide in each instance to insure the dependable and general application of the accompanying formula to all pipes of that class. It is to be hoped that Barnes' values will be checked or revised by other investigators as more data become available, in order that certainty will accompany the use of his constants. A study of all the constants obtained by Barnes for conduits and channels of different material gives rise to the following curious coincidence. If the average value of each constant be calculated and inserted in the general formula there is found to result

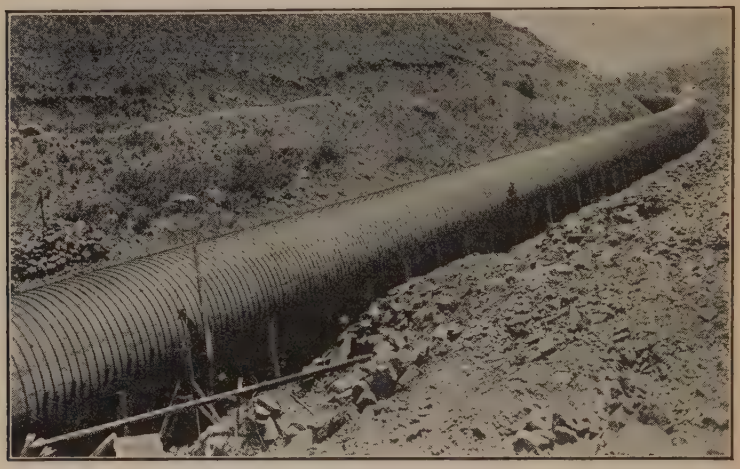
$$V = 144.5\, R^{.64} S^{.52}$$

or

$$V = 110\, R^{.64} S^{.52}\, 0.001^{-.04}$$

which is very closely the Williams-Hazen formula for pipes of average smoothness.

107. Special Formulas for Wood-Stave Pipes.—In 1923 a special study of all available experiments on this kind of pipe was made by Mr. Rodriguez of the Mass. Institute of Technology. The well known data obtained by Messrs. Scobey, Morritz, Moore and Noble were included. It was found that all the data



12-Foot Pipe Line of the California-Oregon Power Co. (Courtesy of Continental Pipe Mfg. Co.)

would be accurately represented if the pipes were divided into two classes,—smooth and very smooth. There was obtained,

For Smooth Wood-Stave Pipe,

$$v = 200\, R^{.66} S^{.59}. \quad . \quad . \quad . \quad . \quad (123)$$

For Very Smooth Wood-Stave Pipe,

$$v = 248\, R^{.66} S^{.59}. \quad . \quad . \quad . \quad . \quad (124)$$

While the exponents are identical with those derived by Barnes for this kind of pipe the coefficients are probably much more accurate than his, representing the data of nearly 100 experiments. The values of f tabulated on page 158 were computed from the above formulas.

108. Schoder's Formulas.—Professor Ernest W. Schoder of Cornell University, whose work as an experimental investigator of pipe-flow phenomena is well known, has proposed the following formulas:

For Extremely Smooth Pipes.

$$v = 277\ R^{.71} S^{.57} \text{ or } h_f = 0.0003 \frac{l}{d^{1.25}} v^{1.75}$$

For Fairly Smooth Pipes,—

$$v = 175\ R^{.67} S^{.54} \text{ or } h_f = 0.00038 \frac{l}{d^{1.25}} v^{1.86}$$

For Rough Pipes,

$$v = 120\ R^{.64} S^{.51} \text{ or } h_f = 0.00050 \frac{l}{d^{1.25}} v^{1.95}$$

For Extremely Rough Pipes,

$$v = 91\ R^{.62} S^{.50} \text{ or } h_f = 0.00069 \frac{l}{d^{1.25}} v^{2}$$

As an aid in selecting the proper formula, Schoder offers the following notes on the several classifications of roughness:

"EXTREMELY SMOOTH PIPES":—New seamless-drawn brass, block-tin and lead, glass, porcelain-like glazed pipes; all with interior surfaces both appearing very even to the eye and feeling very firm and smooth to the touch.

Intermediate between the above category and the one below are all sorts of *newly-laid* so-called "smooth" common commercial pipes such as coated cast-iron, wrought-iron, and wood-stave. High grade rubber-lined fire hose causes about one-third less loss of head than the following category.

"FAIRLY SMOOTH PIPES":—All ordinary pipes after a few (say about five, more or less) years in ordinary service, such as asphalt-coated cast-iron and spiral-riveted steel pipes (latter of thin metal and with very flat rivet heads), wrought-iron, both "black" and galvanized, (but the latter in the small sizes may be "rough pipes" even when new), wood-stave, reinforced-concrete, galvanized spiral-riveted steel. This category is rough enough to be called fairly conservative for general water supply designing purposes.

Intermediate between the above category and the one below are the above-mentioned pipes after being fairly long (say about ten years or so) in service and subjected to average deterioration. Unlined linen "mill fire hose" causes about one-third more loss of head than the previous category of "fairly smoth pipes."

"ROUGH PIPES":—Originally "fairly smooth pipes" that have deteriorated fairly rapidly for some ten or fifteen years after being laid;

also ordinary lap-riveted steel pipes some years in service; also large well-laid brick storm-water sewers flowing full. This category represents a roughness such that its use in design is quite conservative in cases where full capacity will probably not be demanded for some dozen years after laying.

Intermediate between the above category and the one below are pipes having more local roughness or more frequent joints than ordinary water pipes, *e. g.*, ordinary glazed clay sewer pipes in average good service condition, also small brick lined sewers, also small riveted steel pipes made of sections only some two or three diameters long.

"EXTREMELY ROUGH PIPES":—This category represents a degree of roughness or deterioration beyond anything that would ordinarily be allowed for in design of water pipes, say the condition of small street mains after some thirty or forty years of service. In this category come small sewer pipes considerably fouled by slime and deposits or laid with poor alignment.

The above formulas all assume that the exponent of d is constant at 1.25 irrespective of the pipe's roughness. They are necessarily approximate in their application as a particular pipe may rightly have a classification intermediate between those given. Used in conjunction with other formulas, they should be of great value.

109. General.—It is impossible to say which of the various formulas presented in the foregoing articles are to be preferred as accurately representing facts. Each has its proponents and certain advantages, and their lack of close agreement should not confuse the student. In designing a pipe line, one of the prime requisites is that it should have sufficient capacity even after years of service. Deterioration in this respect must be carefully estimated and good judgment, based on long experience, is often more important than an exact knowledge of the formula which represents the pipe's performance when new. Often the recorded flow data of some special pipe may be of aid, and the references at the end of this chapter will be found of great help.

110. Branching Pipes.—In any water supply, water power or sewerage system it is very common to find pipes arranged with branches or laterals which may or may not return to the parent main. Many of the problems thus presented are very complicated and often impossible of exact solution, but certain general princi-

ples may be laid down as applicable in nearly all cases. An understanding of these may be had from a study of the following problems.

(1) *The Problem of Three Reservoirs.*—Fig. 97 illustrates a problem which may arise in the design of water supply systems. A high-level reservoir is to be connected with two others at lower levels by means of a main and branches. Given the length and diameter of each pipe and the levels of the junction and the reservoirs, it is desired to find the rate of discharge between reservoirs. It was pointed out in Art. 99, equation (114), that in the case of

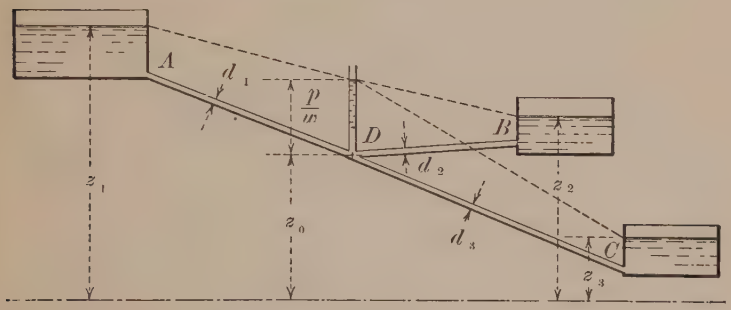


FIG. 97

long pipes the loss by friction is so nearly the entire head h, on the discharging end, as to permit the law of flow to be represented by

$$h = f \frac{l}{d} \frac{v^2}{2g},$$

without incurring an error larger than is apt to result from the selection of a proper value for f.

We may therefore apply this equation to the flow in the pipes AD, DB, and DC, obtaining respectively,

$$z_1 - \left(z_0 + \frac{p}{w}\right) = f \frac{l_1}{d_1} \frac{v_1^2}{2g}, \quad \ldots \quad (1)$$

$$\left(z_0 + \frac{p}{w}\right) - z_2 = f \frac{l_2}{d_2} \frac{v_2^2}{2g}, \quad \ldots \quad (2)$$

$$\left(z_0 + \frac{p}{w}\right) - z_3 = f \frac{l_3}{d_3} \frac{v_3^2}{2g}. \quad \ldots \quad (3)$$

Since there are four unknowns appearing in these equations, $\frac{p}{w}$, v_1, v_2, and v_3, a fourth equation must be had for solution and this is furnished from the *equation of continuity*,—

$$a_1 v_1 = a_2 v_2 + a_3 v_3. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The solution involves the use of tentative values for f, which may be more closely found from the resulting values of v_1, v_2, and v_3.

The addition of (1) and (2) results in

$$z_1 - z_2 = f \frac{l_1}{d_1} \frac{v_1^2}{2g} + f \frac{l_2}{d_2} \frac{v_2^2}{2g}, \quad . \quad . \quad . \quad (5)$$

Similarly the addition of (1) and (3) gives

$$z_1 - z_3 = f \frac{l_1}{d_1} \frac{v_1^2}{2g} + f \frac{l_3}{d_3} \frac{v_3^2}{2g}, \quad . \quad . \quad . \quad (6)$$

which with (4) and (5) enables the flow in each pipe to be found.

It is to be noted that the above solution depends on the assumption that the flow is from reservoir A to both B and C. A little thought will enable the student to see that the dimensions of the pipes and the levels of the reservoirs might be so arranged that the intermediate reservoir at B would discharge through DB instead of being filled by it. If the pipe DB is to *deliver* water to the reservoir B, its gradient must slope in that direction, and this requires that the pressure-head at the junction be such that $\left(z_0 + \frac{p}{w}\right) > z_2$. If $\left(z_0 + \frac{p}{w}\right)$ just equals z_2, then the water in DB will have no movement. If $\left(z_0 + \frac{p}{w}\right) < z_2$ the flow must be from B to D. In any problem, therefore, where the level of B is higher than the junction, it is necessary to first determine the direction of flow in BD. This may be done by assuming that $\left(z_0 + \frac{p}{w}\right)$ just equals z_2, so that in in this pipe no flow occurs. We may then compute, for the resulting value of $\frac{p}{w}$, what the quantities Q_1 and Q_3 would be [equation (1) and (3)]. If Q_3 is found larger than Q_1, it means that flow must be from B to D. For, assuming the reverse to be true, the grade line for DB would slope as shown in Fig. 97, under which condition the actual value of $\frac{p}{w}$, would be greater than assumed, and Q_3

would be increased while Q_1 would be diminished. With Q_3 larger than Q_1 by the *first* assumption, and the difference in amount increased by the *second*, it can be seen that the second is not possible, since it implies $Q_1 = Q_2 + Q_3$. Similarly, if on the first assumption Q_1 is found larger than Q_3, it means that the flow is from D to B. Having thus determined the direction of flow in DB, the discharge from DB and DC may be obtained by the use of the equations previously given. A method which seems preferable to this, however, because of its shortness, is as follows: The direction of flow having been determined, a value for $\frac{p}{w}$ may be *assumed* that will give a slope to the grade line of DB consistent with the found direction. Equations (1), (2), and (3) then serve to give Q_1, Q_2, and Q_3, and Q_1 should equal $Q_2 + Q_3$, or $Q_1 + Q_2 = Q_3$, according as the flow is into or from B. Assuming that Q_1 should equal $Q_2 + Q_3$, but is found to be greater than $Q_2 + Q_3$, then the value of $\frac{p}{w}$ has been assumed too small and must be modified to give the desired relation. Although seemingly a laborious method, it will be found that two or three trials will give a satisfactory solution and the time required will be less than that required to solve equations (4), (5), and (6) simultaneously.

This problem may be extended to any number of reservoirs and solved by this last method.

(2) *Problem of Divided Flow.* One phase of the problem is presented in Fig. 98 which shows a high-level reservoir connected

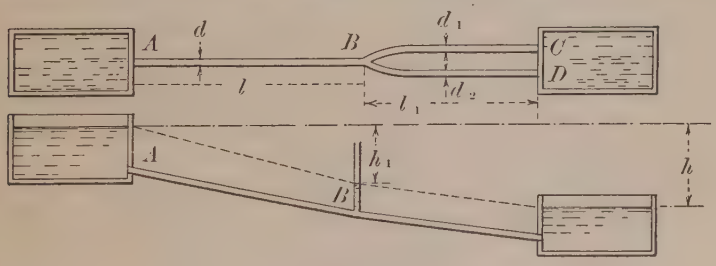


FIG. 98

with a lower one by means of a pipe line whose lower portion consists of two pipes, of diameters d_1 and d_2, laid side by side and entering the second reservoir at the same level. Given the diameters and lengths of the pipes, also the difference in reservoir

level, it is desired to find the rate of flow in the separate pipes of the system. Proceeding, as in the previous problem, with the assumption that the fall in the gradient in a length of pipe is the measure of the head lost in that length, we have for pipe AB,

$$h_1 = f\frac{l}{d}\frac{v^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and for BC

$$h - h_1 = f\frac{l_1}{d_1}\frac{v_1^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

while for BD

$$h - h_1 = f\frac{l_1}{d_2}\frac{v_2^2}{2g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The addition of (1) and (2) results in

$$h = f\frac{l}{d}\frac{v^2}{2g} + f\frac{l_1}{d_1}\frac{v_1^2}{2g},$$

and (1) and (3) give

$$h = f\frac{l}{d}\frac{v^2}{2g} + f\frac{l_1}{d_2}\frac{v_2^2}{2g}.$$

These two equations contain three unknown velocities, but a third equation is furnished by the relation

$$av = a_1v_1 + a_2v_2.$$

The solution is then possible.

It should be noted that the head lost by friction in BC is the same as in BD [see (2) and (3)]. If the total loss in head between reservoirs be desired, it will be the loss in AB plus the loss in *one* of the branch pipes. In case it is difficult for the student to see that this is so, let him bear in mind that *lost head* and *lost energy per pound* of water flowing are synonymous terms (Art. 39) and that the *total* lost energy per pound will be the sum of the losses in the main pipe and *one* of the branches.

If the branch pipes have the same diameter, then only two velocities are unknown and two equations will suffice. These are

$$h = f\frac{l}{d}\frac{v^2}{2g} + f\frac{l_1}{d_1}\frac{v_1^2}{2g},$$

and

$$av = 2a_1v_1.$$

(3) *Problem of Divided Flow.*—A common arrangement of piping is that shown in Fig. 99, where, for purpose of distribution, a main

line is tapped by a smaller pipe which later returns to the main. Assuming that no water is drawn from the main or branch between B and E and that the draught at F, and beyond, causes the grade line at that point to lie at a distance H below the reservoir level, we may proceed to find the velocities of flow in the different pipes as follows. Since the pressures at B and E may be assumed as common to both pipes BCE and BDE, and therefore the total fall in

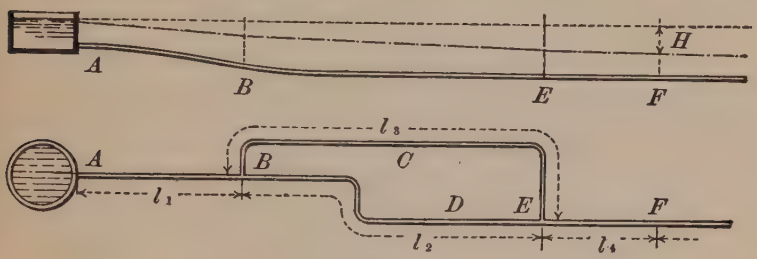


FIG. 99

the gradients of these pipes the same in amount, it follows that the lost heads in the pipes are equal, or

$$f_2 \frac{l_2}{d_2} \frac{v_2^2}{2g} = f_3 \frac{l_3}{d_3} \frac{v_3^2}{2g}.$$

As for the total loss in head between A and F we have

$$H = f_1 \frac{l_1}{d_1} \frac{v_1^2}{2g} + f_2 \frac{l_2}{d_2} \frac{v_2^2}{2g} + f_4 \frac{l_4}{d_4} \frac{v_4^2}{2g}. \quad . \quad . \quad . \quad (a)$$

since, as in the previous problem, we may include the loss in but *one* pipe between B and E. To obtain v_1, v_2 and v_4 from this equation, it is necessary to have other equations, and these are furnished by the relations

$$a_1 v_1 = a_4 v_4, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

$$a_1 v_1 = a_2 v_2 + a_3 v_3, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

and

$$f_2 \frac{l_2}{d^2} \frac{v_2^2}{2g} = f_3 \frac{l_3}{d_3} \frac{v_3^2}{2g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

The solution of these four equations simultaneously gives the desired velocities. Using the relations given in (b), (c), and (d), equation (a) may be written

$$H = \left[c_1 + c_2 c_3 \left(\frac{a_1}{a_2 \sqrt{c_3} + a_3 \sqrt{c_2}} \right)^2 + c_4 \left(\frac{a_1}{a_4} \right)^2 \right] \frac{v_1^2}{2g},$$

in which c_1, c_2, c_3, and c_4 have replaced the more cumbersome ratios,

$$f_1 \frac{l_1}{d_1}, f_2 \frac{l_2}{d_2}, \text{ etc.}$$

In the first solution of this equation it is necessary that an average value for the different friction factors be assumed and afterward corrected according to the resulting velocities. The algebraic work involved in the above solution is somewhat laborious and when it becomes necessary to investigate the rates of flow for different values of H, it will be found much easier to use a graphical solution which is to be found in Art. 111.

(4) *Main Line with Laterals.*—Whenever a pipe line becomes the main of a distributing system, the laterals of which lead from the main at irregular intervals along its length, a very complicated problem arises in computing the necessary size for the main so that the pressure will not fall at any point below a stated amount.

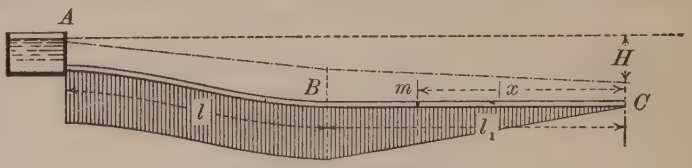


FIG. 100

A solution anything more than approximate is impossible unless all the data concerning the sizes of the laterals and the amounts of their maximum simultaneous withdrawals are known. As these are generally unknown, the following rude approximation may be made. Figure 100 shows such a main line supplied with water from a reservoir A. In the upper portion AB of the main there are no laterals, and the assumption will be made that from the point B to the end of the pipe at C the laterals are spaced equidistant on the main and draw equal quantities of water from it. At C the main is dead-ended. The assumption of equal draught by the laterals causes a uniform reduction in velocity in the main between B and C, the velocity at C being zero. In the portion AB the velocity is v and is constant. Below the pipe is sketched a shaded area to represent the variation in velocity in the main. It will be seen that at any point m, distant x feet from C, the velocity is

$$v_m = v \frac{x}{l_1}.$$

The lost head occurring in a dx length of pipe at this point would be

$$\text{L. H.} = f \frac{dx}{d} \frac{v^2}{l_1^2} \frac{x^2}{2g}.$$

To find the total head lost between B and C we may integrate the above expression with x a variable whose limits are zero and l_1. There results,

$$\text{Head lost in } BC = \frac{1}{3}\left(f \frac{l_1}{d} \frac{v^2}{2g}\right),$$

or the amount lost is one third that lost in an equal length of main containing no laterals. For the head lost in the entire main we have

$$\text{L. H.} = f \frac{l}{d} \frac{v^2}{2g} + f \frac{l_1}{3d} \frac{v^2}{2g}.$$

It will be noted that f is not constant for the total portion BC and whatever value is assumed for it will therefore be an approximation. In actual design, however, this need cause no serious error, since liberal allowance has to be made in choosing a value for f to provide for deterioration of the pipe. The lost head as given by the last equation may be subtracted from the static head on the end of the pipe to give the pressure-head at that point. Since the general requirement is to find the diameter of pipe necessary for fixed values of pressure-head at C and quantity Q, we may substitute for v its equal, $\frac{4\ Q}{\pi d^2}$, obtaining for d

$$d = \sqrt[5]{\left(l + \frac{l_1}{3}\right) \frac{16\, fQ^2}{2\, gH\pi^2}}.$$

As stated above, results obtained by the process of reasoning just given will be approximate only and should be modified to meet the peculiar conditions of each problem.

111. Graphical Solution for Problems in Divided Flow.—In the design of municipal distributing systems, the problem of divided flow, as illustrated under paragraph (3) of the previous article, often presents itself. The flow to any one point is generally not by a single pipe route, but by means of two or even more routes, and the question arises as to the amount which can be furnished at the point without excessive loss in head. Thus in Fig. 99 it

might be desired to find what pressure could be obtained at the point F under certain specified rates of flow. Freeman has suggested a graphical method of handling the problem which affects a considerable saving in time and labor, especially if several different rates of flow are to be investigated.

The loss in the pipes AB and EF may be easily computed and tabulated for the specified rates of flow. For the divided flow between B and E it will be convenient to plot two curves, BCE and BDE (Fig. 101), each showing the relation between the head

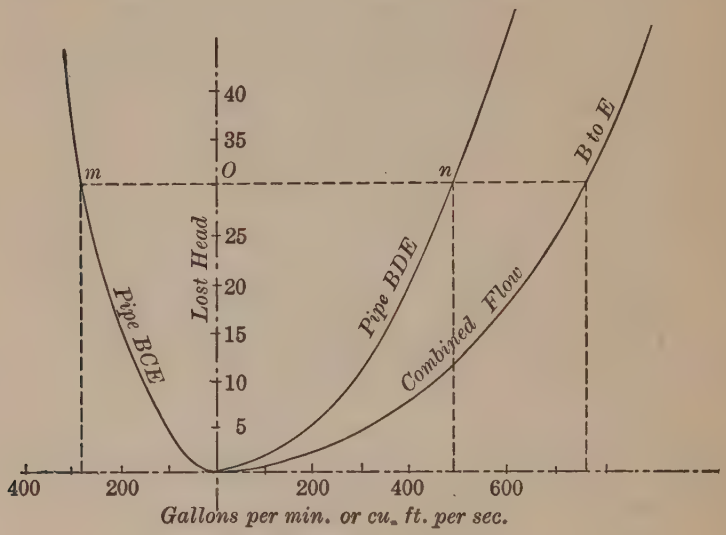


FIG. 101

lost in one of the pipes and the rate of flow in the pipe, as the latter is made to vary. Each curve may be easily constructed from the plotting of a few points showing the head lost at arbitrarily assumed rates of flow. It should be borne in mind, however, that the rate of flow in either pipe will be less than the combined flow and hence rates that are less than the desired combined flow should be assumed. In the plot as made, the two curves have been drawn with the rates of flow laid off on opposite sides of the origin and it will at once be noted that any horizontal line, m-n, will represent by its length, the combined flow of the two pipes when the head lost is that as intercepted by the line on the axis of ordinates. Also m-O will represent the flow in BCE, and O-n the flow in BDE. A

line representing any desired rate of flow may be similarly fitted to the curves and the corresponding lost head determined. If desired, a third curve, as shown, may be plotted to give directly the relation between head lost and the corresponding combined flow.

It will be seen that any number of pipes may run from B to E and the relation between flow and loss determined in the above manner. Where successive diversions are met with it will be found that the graphical method is the only practical one. A description of the method by Freeman, in which he shows its applicability to more complicated problems, may be found in the Journal of the New England Waterworks Association for Sept., 1892.

112. Pipe Line with Nozzle.—The combination of a nozzle and pipe line is very common in engineering. It is used in fire

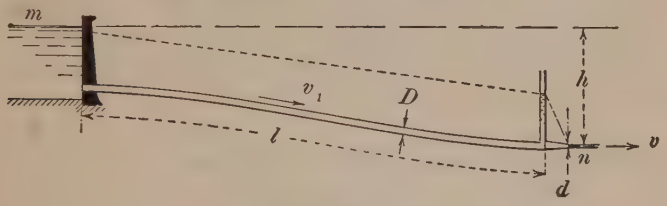


FIG. 102

fighting, mining and power developments. Fig. 102 illustrates a general case, the water being supplied to the pipe from a reservoir.

If we apply Bernoulli's Theorem to the points m and n, we obtain

$$h = \frac{v^2}{2g} + 0.5\frac{v_1^2}{2g} + f\frac{l}{D}\frac{v_1^2}{2g} + \left(\frac{1}{c^2} - 1\right)\frac{v^2}{2g}, \quad . \quad (125)$$

the last term representing the head lost in passing through the nozzle (Art. 59). In this equation appear two unknowns, v and v_1, but a relation between them being given by

$$\frac{v}{v_1} = \frac{D^2}{d^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (126)$$

a solution for either of them is possible.

In power developments where the kinetic energy of the jet is utilized by a turbine, it often happens that more than one nozzle is attached to the pipe line, furnishing power to several wheels or even to the same wheel. It is worthy of note that equation (125)

would apply in such a case, and the total loss in head would consist of pipe losses and the loss in *one* nozzle. The latter statement may be checked by remembering that lost head is synonymous with lost energy *per pound* (Art. 39). Equation (126) must, however, be re-written since the discharge is now through *two* nozzles. If the nozzles are of equal area,

$$2\,av = a_1\,v_1$$

or

$$\frac{v}{v_1} = \frac{D^2}{2d^2}.$$

The amount of energy furnished by a jet is seen to largely depend upon the energy lost in the pipe line, inasmuch as the loss in the nozzle is relatively small. In power work it is therefore essential to use as large a pipe as possible in combination with a given nozzle in order to reduce the loss by pipe friction. There is an economical limit, however, to the size which may be used, which will be reached when further increase in diameter causes an additional outlay, the interest of which will exceed the return from the increase in power.

113. Variation in Velocity at a Cross-Section.—As pointed out in Art. 34 and again in Art. 89, the velocities of all particles passing a given cross-section of a pipe are not alike either in direction or magnitude, due to the sinuosity of the stream lines. Neither is there a general relation between the velocities except that their axial or *forward* components have a maximum value at or near the center of the pipe and decrease markedly to a minimum as the side is approached. This is due to the friction at the side and between adjacent water particles. Much experimental work has been done to ascertain the law which governs this latter variance, but so far it has not been definitely established. The usual method of procedure has been to use Pitot tubes (Art. 114) to measure the forward components at various points along a diameter, and plot to scale these velocities from a reference line, *ab* (Fig. 103). Through the plotted points a smooth curve, *cde*, is then drawn and, if possible the nature and equation of the curve

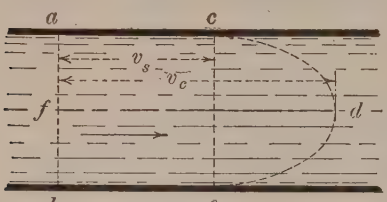


FIG. 103

determined. The practical difficulties encountered are the irregularities in the plotted points, and the impossibility of making a measurement at or near the side of the pipe where the curve is changing rapidly. Careful experiments by Bazin and by Williams, Hubbell and Fenkell, indicate that an ellipse can be made to fit the position of the plotted points more nearly than any other mathematical curve. The major axis of the ellipse will be coincident with the axis of the pipe and the curve will be tangent or nearly so to the pipe walls. Bazin worked with a cement pipe 2.73 feet in diameter, and Williams, Hubbell and Fenkell made their measurements on cast iron water mains in Detroit having diameters of 12, 16, 30 and 42 inches. Assuming an ellipse to well represent the variation in velocity, we may deduce a relation between the center velocity, v_c, and the mean velocity, v_m, for the entire section.

From Fig. 103, it may be seen that the amount of water passing the section per second is represented by the solid generated by rotating the figure *acdeb* about *fd* as an axis. Its volume may be divided into two parts,—one a cylinder *aceb*, and the other an ellipsoid *cde*. The volume of the cylinder is $\pi r^2 v_s$ (v_s being the side velocity) and of the ellipsoid, $2 \pi r^2 \frac{(v_c - v_s)}{3}$, since the volume of an ellipsoid is two-thirds that of the circumscribing cylinder. If we divide the total volume of the solid by the area of the pipe we obtain

$$v_m = \frac{\pi r^2 v_s + 2 \pi r^2 \frac{(v_c - v_s)}{3}}{\pi r^2} = \frac{v_s + 2 v_c}{3}.$$

The experiments above mentioned indicate that the value of the side velocity is about one half that at the center, when the pipe is straight for a considerable distance. If we assume

$$v_c = 2 v_s,$$

we obtain

$$v_m = 0.83 v_c \quad . \quad . \quad . \quad . \quad (127)$$

The Detroit experiments indicated the relation $v_m = 0.84 v_c$ and this value was found to hold good for all sizes of pipe used in the experiments. Other experiments on pipes of small size seem to show a slight decrease in the coefficient with decrease in diameter, but this is not certain. The coefficient, or ratio of v_m to v_c, is often spoken of as the *Pipe Coefficient.*

In passing a bend in the pipe, it is found that a rearrangement of the velocities takes place, the thread of maximum velocity being found nearer the outside wall of the bend; and a considerable length of pipe will be traversed by the water before resuming normal arrangement of velocities. Similarly any disturbing factor such as an enlargement or contraction in the pipe will for the moment cause a redistribution of velocities that will persist throughout many diameters of length. It is highly probable that, due to interlacing of stream lines, particles of water are continually moving from the center of the pipe toward the sides and back again, changing continually the forward component of their velocity. In this way it would be possible for particles that simultaneously crossed a given section to be together at a later section, even though they had crossed the first section with different velocities and at different distances from the center. This is strongly indicated in the experiments of Prof. C. M. Allen upon the measurement of flow in pipes by the Salt-velocity Method (see Art. 117).

114. The Pitot Tube.—This device, invented by Pitot in 1732, is essentially an instrument for measuring the velocity at any point in a moving stream. It was developed originally for use in open channels and jets, but it was later found to be of greater service in the case of streams confined under pressure, as in pipes and conduits. For this reason it is introduced and discussed here with other pipe-flow measuring devices, rather than in an earlier article.

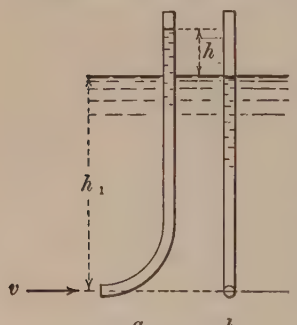


FIG. 104

In its simplest form and one adaptable to open channels and jets, it consists of a small tube having its lower end bent through 90°, so that it may be placed in the stream with its open end turned directly against the current, as shown in Fig. 104*a*. The dynamic pressure against the orifice causes a head h to be maintained above the free surface of the water. If the tube be turned so that the plane of the orifice is parallel to the direction of the stream lines, the pressure on the orifice is static only, and the level of the water in the tube is very closely that of the free stream surface (Fig. 104*b*). The

relation between h and v for the first position is that ordinarily stated between head and velocity,

$$h = \frac{v^2}{2g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (128)$$

Although a strict analytical proof for this is not possible, it seems logical to believe that, inasmuch as a head h will produce a velocity v, a stream with velocity v will maintain a head h by reason of the dynamic pressure it can furnish. Ample experimental proof of the above relation is furnished by the experiments of W. M. White, published in the *Journal of the Association of Engineering Societies*, August, 1901. Mr. White caused a vertical jet, from an orifice located in the horizontal floor of a reservoir, to fall upon a flat circular plate placed at right angles to the jet (Fig. 105). Through the plate were drilled numerous holes, each one being connected with an open piezometer placed vertically. One of the latter is shown in the figure, connected with the hole that is under the central portion of the jet. In this piezometer the water stood at practically the same elevation as in the tank, showing that the dynamic pressure produced on the plate at the point where the piezometer was attached was just sufficient to sustain a column of water to a height equal to that through which the water in the jet had fallen. Since this height was the velocity head in the jet at the point where it met the plate, the truth of the above-stated relation seems apparent.

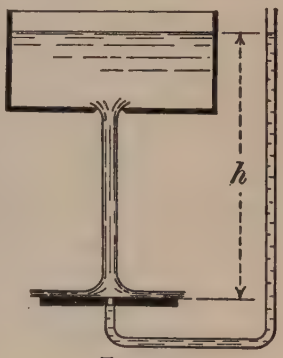


FIG. 105

If Bernoulli's Theorem be written between a point in the moving stream just ahead of the tube's point, and a point at the top of the water column in the tube, the relation given in (128) will be obtained; but, inasmuch as a particle of water passing the first point never reaches the second point (*i. e.*, no stream-line flow), it is apparent that the theorem does not apply.

Tubes of the form shown in Fig. 104 are not of much service in ordinary streams as it is difficult to measure the low velocities with the desired accuracy. Thus a velocity of 1 ft. per second would be indicated by a column h, less than $\frac{1}{5}$ of an inch in height,

and velocities of this size and less are very common in open streams. For high velocities, especially in jets, it is very useful.

Darcy improved the tube by making its tip conically convergent and adding a second tube having its point directed downstream (Fig. 106). In this the water stood at a lower level than the surface of the stream, and the difference h between the levels in the two tubes was used to determine v. For this form of tube it was necessary to write.

$$v = c\sqrt{2gh}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (129)$$

and to determine c by an experimental rating of the tube. This was done, either by drawing the instrument through still water at known velocities, or placing it in streams whose velocity at the point of insertion was known. The latter method has been shown to be the preferable one, inasmuch as the first condition is one unlike that under which the tube is used. In still water the point of the tube moves through water particles which are at rest and the effect of straight-line flow is produced, whereas under working conditions the flow of water to and by the tube is sinuous or turbulent. Fig. 106 shows the two tubes joined at their upper ends and fitted with an air-cock, a. This can be closed after the water in the tubes has found its levels, and the whole apparatus removed from the stream to more carefully measure h. A modification of Darcy's tube had the second point turned until its orifice lay in a plane parallel to the stream's motion. Theoretically the relation, $v = \sqrt{2gh}$ should have held good for this tube, but it was found that a coefficient was necessary as in equation (129). The coefficient varied widely with the construction of the tubes, and it was thought at the time that the form of the velocity point was largely responsible for this. It is now known that the construction of the tube affected the stream-line flow past the second, or pressure, opening and so affected the pressure at the opening and therefore the head in the second tube.

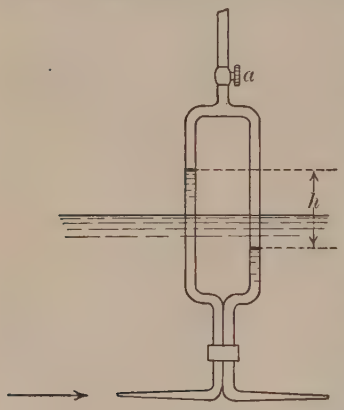


FIG. 106

In 1888, Freeman used the simpler Pitot tube to measure the velocities in the different parts of the cross-section of jets from fire nozzles, and there it gave results which were highly satisfactory. The orifice of his tube was formed from the point of a stylographic pen, having an opening of about 0.006 of an inch in diameter.

The Pitot Tube in Pipes.—Experiments in this country and abroad have shown that Pitot's tube is well adapted to determining the velocity at any point in a closed pipe. Since, in this case, the velocity column has its height affected by the pressure in the pipe, it is necessary that a second tube be used to measure the pressure-head. From the observed head in the first tube may be subtracted the pressure-head to obtain that due to velocity alone.

Williams, Hubbell, and Fenkell used Pitot tubes, designed on this principle, in their well-known experimental work on the water mains of Detroit, in 1898, and developed a very ingenious and practical form of tube which could be inserted through the side of a pipe by means of a stuffing box tapped into it. Through the stuffing the tube was free to move, thus allowing its orifice to be set at any desired point on the pipe's diameter. In order to more accurately measure the difference in head in the two tubes, they connected the water columns with a differential oil gauge of the type described in Article 15. For their form of tube it was found that the velocity at the point of the impact opening was given by the relation

$$v = c\sqrt{2gh},$$

c having a value less than unity. The value of c was established by rating the tube both in still and moving water as explained previously in this article. It was found that the two methods gave slightly different values and indicated that it is essential to rate tubes under service conditions.

In 1901–2 experiments were conducted by Gregory, Maltby and White with a view to determining what effect various form of impact and pressure openings have upon the reduction coefficient c, and to see if it were not possible to design a tube that should have a coefficient of unity.

Without relating in detail the various experiments, references to which appear at the end of this chapter, it may be said that Mr. White proved conclusively that the form of the impact point has little or no effect on the coefficient of the tube, and that velocity

head is changed into pressure-head according to the law $v = \sqrt{2gh}$, *provided* the impact surface surrounding the point is a surface of revolution having its axis parallel to the direction of the stream's motion. Upon the form of the pressure opening will depend the amount by which c varies from unity. To obtain a pressure column in the second tube, truly indicative of the pressure at the tip of the impact opening, the pressure and impact openings must not only be near together, but the former must be constructed in accordance with the laws of piezometer openings as established by Mills and stated in Art. 38. The pressure at the pressure opening must not be lowered by eddyings or deflected water, caused by the presence of the impact tube or the construction of the pressure tube itself, as this will result in a value for c of less than unity. Messrs. Gregory and Maltby succeeded in designing a tube which, judged from their experiments, seems to have the desired constant coefficient of unity. At its lower end it consists of a very small tube, which is brazed into the larger one, forming the impact or velocity opening (Fig. 107). The small tube runs through the larger to its top and there connects with a pressure gauge, preferably of the oil differential type. The outer tube has three small openings in its side, spaced about equally around the circumference, forming the openings for the pressure-head. The top of the outer tube is also connected to the differential gauge.

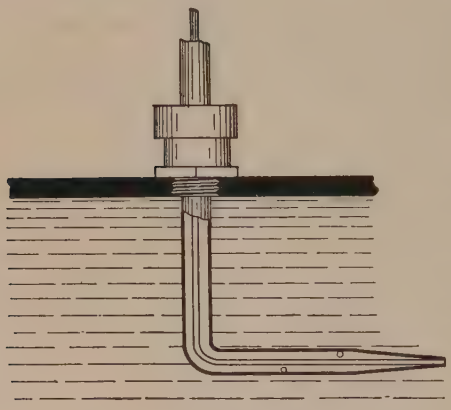


Fig. 107

Of the many forms experimented with, this tube alone seemed to possess the desired properties, and it was clearly shown that the usual variance in coefficient was due to incorrect design of the pressure openings rather than to the shape of the impact point.

When properly made and used, the Pitot tube is a most valuable device for measuring velocities in closed conduits and, in the hands of experts, should give results highly accurate. To insure this,

any form of tube should be rated under service conditions in order that the value of the coefficient be definitely established.

In the following paragraph, the method of using a tube to determine the mean velocity and rate of flow in a pipe is outlined and a description given of the Cole Pitometer which is a commercial form of Pitot tube.

115. Determination of Mean Velocity and Rate of Discharge. Traversing.—The variation in velocity across a section of pipe has already been discussed in Art. 113. To determine the approximate mean of the various velocities by using a Pitot tube, it is customary to insert the tube on a diameter and successively note the deflections of the attached gauge columns as the tube occupies different points on the diameter. From these readings the value of each corresponding velocity can be computed. If the points occupied be well distributed across the diameter, the mean velocity may be assumed equal to the arithmetical mean of those measured. A more accurate method is to subdivide the pipe's section into equal areas by means of concentric rings and to determine the mean velocity of each ring by observations made at the middle of each ring (Fig. 108). Denoting the areas and corresponding velocities by the numbers of the rings, the discharge past the section will be closely equal to

FIG. 108

$$Q = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5$$

or, since the areas are all equal,

$$Q = a\,(\Sigma\, v).$$

For the mean velocity,

$$V_m = \frac{Q}{A} = \frac{a\,(\Sigma\, v)}{A} = \frac{\Sigma\, v}{n}$$

n being the number of rings into which the section is divided. Sometimes the measurements are continued across the diameter, two determinations of velocity being made in each ring. In this

case the point of the tube is stopped twice while traversing the center area at points dividing the diameters of this area into thirds. When double measurements are thus taken in all rings we have

$$Q = a\left(\frac{\Sigma v}{2}\right)$$

and

$$v_m = \frac{\Sigma v}{2n}$$

The object to dividing the pipe's section into rings of *equal area* is to give the observed velocities equal weight so that their arithmetical mean will more closely be the *true* mean.

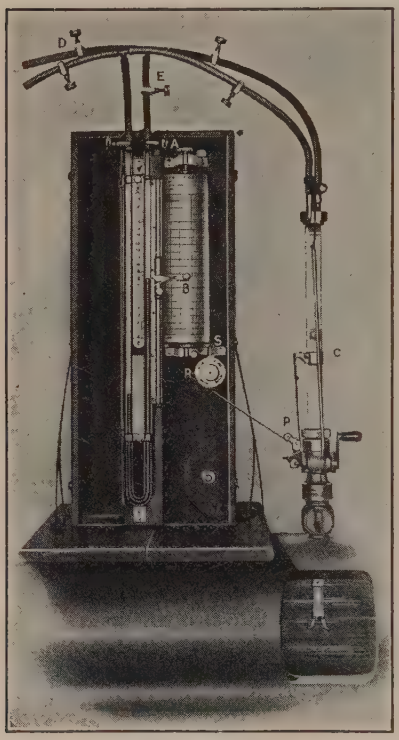


FIG. 109

Mr. Cole has devised a tube for use in pipes which is an adaptation of Darcy's double-pointed tube. It is commercially known as the Pitometer, being manufactured by the Pitometer Company of New York City. A general view of it is given in Fig. 109. Its construction permits its being inserted into a pipe through a standard corporation cock without shutting off the water from the pipe. Flexible tubing connects the impact and pressure tubes with a differential gauge, and provision is made for registering the gauge readings on a revolving paper as the tip of the instrument is moved across the diameter. Its coefficient has been determined by the makers to be 0.84.

116. The Venturi Meter.—This device, invented by Herschel in 1887 and named by him after the distinguished philosopher who first experimented with diverging tubes, is designed to measure the *rate* of flow in pipe lines. It is extremely simple in form and

detail, consisting merely of two conically converging and diverging sections of pipe joined by a short cylindrical section as shown in Figs. 110 and 111. The whole is inserted in the pipe line whose

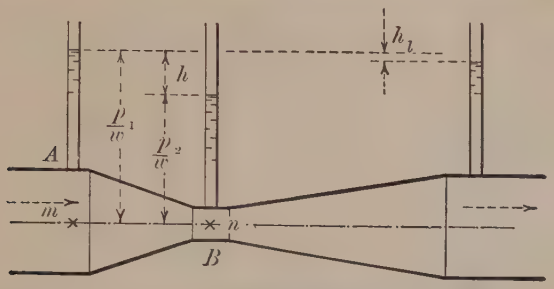


FIG. 110

flow is to be metered. Pressure gauges fitted to the main pipe at A and to the throat of the meter at B, measure the pressure-heads at these points and complete the apparatus in its essentials. Flow takes place in the direction indicated by the arrows and the

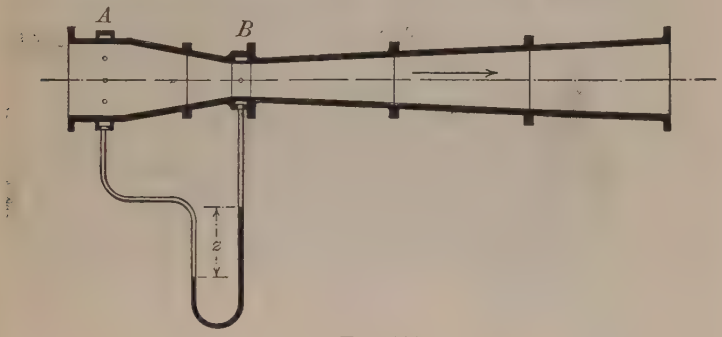


FIG. 111

meter is a very practical illustration of the principles embodied in Bernoulli's Theorem. Between the points m and n (meter horizontal) we may write

$$\frac{v_1^2}{2g} + \frac{p_1}{w} = \frac{v_2^2}{2g} + \frac{p_2}{w},$$

if the loss in head between the two points be neglected. If a_1 and a_2 represent the areas of the sections which contain the points, we may also write

$$v_1 = \frac{Q}{a_1} \text{ and } v_2 = \frac{Q}{a_2}.$$

Then values substituted in the first equation give

$$Q = \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \sqrt{2g\left(\frac{p_1}{w} - \frac{p_2}{w}\right)}. \quad . \quad . \quad . \quad (130)$$

This equation having been derived by neglecting the loss in head between m and n, it is an ideal relation only, and for a practical equation we must write,

$$Q = c \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \sqrt{2g\left(\frac{p_1}{w} - \frac{p_2}{w}\right)}. \quad . \quad . \quad . \quad (131)$$

The coefficient, c, is known as the Meter Coefficient and appears to be fairly constant for all sizes of meter as long as the velocity is

115-inch Venturi Meter Installed at Baltimore, Md. (Courtesy of Builders Iron Foundry, Providence, R. I.)

above the Critical Velocity and the general proportions of the meter similar to those adopted by Herschel. From many experiments by Herschel and others on meters having inlet diameters from 2 to 108 inches, it has been found that c ranged from about 0.94 to values greater than unity. The majority of values were between the limits of 0.97 and 0.99, and from a wide experience with the meter, Herschel recommends the use of 0.98 when a value has not been obtained by previous calibration.

Figure 110 shows a sectional view of a Venturi meter as commonly constructed. The converging portion is comparatively short, its length varying from 2 to 2½ times the diameter of the

main pipe. In this portion the loss is therefore very small, depending on the diameter, the material of the meter, and the velocity of flow. The throat of the meter usually has a diameter $\frac{1}{3}$ that of the main pipe, so that the velocity through it is 9 times the pipe velocity. The diverging portion is made *gradually expanding* in order that the loss in head due to decrease in velocity may be small. Previous experiments having indicated that for a given head the discharge from a diverging tube would be a maximum when the angle of divergence was about 5 degrees, Herschel reasoned that a tube having this flare would cause the least loss in head for a given discharge, and adopted this angle for the diverging portion of his meter.

Graphic Recorder for Venturi Meter as Manufactured by Builders' Iron Foundry, Providence, R. I.

For measuring large quantities of water flowing in pipes, the Venturi meter is the simplest method available and sufficiently accurate for most purposes. Where the coefficient has been determined by a previous rating, the probable error in computing the discharge should be less than 1 per cent., provided the involved measurements are precisely made and the water brought to the meter in a normal condition of flow. The writer has in mind an installation of a 30-inch meter in the discharge line of a centrifugal pump at a point about 12 feet distant from the pump. Careful determination of the meter coefficient indicated a varying value having a mean of about 0.82. Investigation revealed that the discharge from the pump was not only turbulent but followed a spiral direction, and it was not until baffle plates had been inserted in the pipe ahead of the meter, that the coefficient assumed a normal value. The placing of a meter too close to a pump discharge, an elbow or other construction which may cause abnormal turbulence, should be avoided.

The meter differs from most others in that it measures *rate* and not *volume*. Where it is desired to keep a continuous record of the rate of flow, the difference in pressure-heads at inlet and throat is registered mechanically by a pencil-graph drawn on paper kept in motion by clockwork.

The head lost in passing the meter is not large and the insertion of the meter in any pipe line is generally unobjectionable. From the experiments of Herschel and others, the writer concludes that the lost head may be approximated by assuming it equal to 0.15 $\frac{v^2}{2g}$, v being the velocity in the throat.

One of the advantages offered by this type of meter is its ability to handle water laden with sediment or impurities, and liquids of different viscosities. Of course, the meter coefficient for each liquid should be determined. Since there are no moving parts in the meter, subjected to wear, the value of the coefficient should remain practically constant.

117. Determination of Discharge by Salt-Velocity Method.— Professor C. M. Allen, of Worcester, Mass., in collaboration with E. A. Taylor, has devised a method for measuring the rate of discharge in pipe lines and conduits which "is based on the fact that salt in solution increases the electrical conductivity of water. Salt solution is introduced near the upper end of the conduit, and the passage of the solution across one or more pairs of electrodes, at other points in the conduit, is recorded graphically by electrical recording instruments. The passage of the salt solution between two points is accurately timed, and the *volume* of the conduit between the same points is accurately determined. The discharge in cubic feet per second equals the above volume in cubic feet divided by the time in seconds."

The quotation is from a paper by Allen & Taylor as presented to the American Society of Mechanical Engineers in December, 1923. The paper contains a detailed account of the development of the method and of the many experiments which the authors made to prove its reliability. It proves conclusively that when properly conducted, the method closely checks the discharge as obtained by weighing, which is the most accurate method known for measuring water.

In applying the method, a concentrated solution of salt and

water is injected into the conduit, preferably by means of a pop-valve. The effect is to produce in the conduit a short slug of the solution, which, as it advances along the conduit, tends to elongate axially due to the frictional resistance offered by the conduit walls. That the timing of the *center of gravity* of the slug as it moves along the pipe affords a basis for determining the rate of flow, Professor Allen has proved by his experiments. The details of the method would require more explanation than the present space allows and the student is referred to Allen's excellent paper. The statement embodied in the last sentence of the quotation appearing above may be algebraically set forth as follows:—

$$Q = \frac{\text{Volume}}{\text{Time}} = \frac{\text{Mean Area} \times \text{Length}}{\text{Time}}$$

From this it will be seen that with a conduit of *constant* area, the mean velocity is the length (between electrodes) divided by the time, as one would expect. The value of Q by the above equation may be obtained for a conduit of changing section provided its volume can be determined.

PROBLEMS

1. Find the rate of discharge from a 30-inch pipe, 3220 ft. long, supplied from a reservoir whose surface is 46 ft. above the pipe's end. Assume a clean, cast-iron pipe and make the computations (1) allowing for loss at entrance, (2) neglecting it.

2. Assuming a 12-inch pipe line, 2400 ft. long, to be made from wood staves, what will be its rate of discharge under a head of 32 ft.? What if the length were only 1200 ft.?

3. Two reservoirs with a difference in level of 90 ft. are connected by 2 mi. of 15-inch cast-iron pipe. Compute the rate of discharge.

Ans. ~~7.2~~ 7.6 cu. ft. per sec.

4. At what rate will water be discharged through 600 ft. of rubber-lined cotton hose, 3 in. in diameter, if it be attached to a hydrant at which the pressure is 60 lb. per square inch during flow? Assume no nozzle on the end of the hose and $f = 0.022$.

If a 1-inch nozzle be used, what discharge may be expected if a pressure of 70 lb. at the hydrant can be maintained? For the nozzle $c_v = c_d = 0.96$? Assume hose horizontal and $f = 0.024$.

Ans. (1) 285 gal. per minute,
(2) 185 gal. per minute.

5. The pressure-head at a point in a 12-inch cast-iron pipe is 50 ft. At a point 1000 ft. beyond, in the direction of flow, the pressure is 20 lb. per square inch. If the discharge be 5 cu. ft. per second, and 2 ft. of head is lost at intervening bends, what is the slope of the pipe?

Ans. 0.011.

6. At a distance of 2000 ft. (measured on pipe) from the supplying reservoir, a 12-inch riveted steel pipe is 140 ft. below the reservoir's surface and the pressure is 50 lb. per square inch. What is the velocity in the pipe? Assume $f = .025$.

Ans. 5.6 ft. per second.

7. An 18-inch cast-iron pipe is discharging 3000 gal. per minute. At a point 1000 ft. from the supplying reservoir (measured on pipe) the center of the pipe is 80 ft. below the reservoir surface. What pressure, in pounds per square inch, is to be expected there?

Ans. 33.4 lb. per sq. in.

8. Through a rusty iron pipe, 6 inches in diameter, the observed velocity is 8 ft. per second. At a section A on the pipe, the measured pressure-head is 89.7 ft., while at B, 100 ft. farther on where the pipe is 3 ft. below A, the pressure-head is found to be 78.0 ft. Compute the probable value of f.

9. Compute the value of the lower critical velocity for a pipe 3 inches in diameter carrying water whose temperature is 70° F.

10. A 6-inch pipe suddenly enlarges to a diameter of 18 in., the velocity of flow in the 18-inch pipe being 2 ft. per second. Compute the lost head in feet and the foot-pounds of energy lost per second. What difference in pressure will be found in the two pipes near the enlargement? What would be the head lost if the velocity in the 18-inch pipe were 0.5 ft. per second?

Ans. (1) 3.98 ft.
(2) 884 ft.-lb.
(3) 0.43 lb. per square inch.
(4) 0.25 ft.

11. A level pipe line is abruptly enlarged from 4 to 8 in. in diameter. The velocity in the 4-inch pipe being 16 ft. per second, how much energy is wasted in heat at the enlargement? If the pressure-head in the 4-inch pipe be 50 ft., what will it be in the 8-inch?

Ans. (1) 195 ft.-lb. per second.
(2) 51.49 ft.

12. A 24-inch pipe is successively reduced in size to 12 in. and 6 in. the reductions being abrupt in each case. With a discharge of 2 cu. ft. per second, what total loss will be occasioned by the reductions? What difference in pressure will there be in the 24- and 6-inch pipes if 2 ft. of head be lost by pipe friction in the 12-inch pipe?

Ans. (1) 0.55 ft.
(2) 1.8 lb. per square inch.

13. Compute the probable drop in pressure as a pipe line carrying 6 cu. ft. per second suddenly changes from 12 inches to 6 inches in diameter.

14. A long pipe line, 12 inches in diameter, contains 15 bends, each approximately through 90°. What allowance in head should be made for the effect of these bends if $Q=6$ *c. f. p. s.*?

15. A gate valve of the type illustrated in Fig. 91*a*, is closed so that $\frac{d'}{d}$ is 0.75. Compute the head lost at the valve when placed in a 36-inch pipe when the velocity of flow is 6 ft. per second. What loss is occasioned by the valve seat when the valve is wide open?

16. Compute the diameter of pipe necessary for discharging 1500 gal. of water per minute, the pipe being of cast iron, 1000 ft. long, and its discharging end 4 ft. lower than the surface of the reservoir supplying it.

Ans. $d=13.5$ in.

17. A pipe line, discharging into air, consists of two sections, one 500 ft. long and 12 inches in diameter, the other 1200ft. long and 18 in. in diameter. If the change in section be abrupt and the quantity discharged be 3 cu. ft. per second, find the loss in head in each section due to pipe friction and the loss due to sudden enlargement. Plot the hydraulic grade line choosing suitable scales. Assume $f=0.02$.

Ans. Lost heads are 2.46 ft., 0.77 ft., and 0.07 ft.

18. From a reservoir whose level is 300 ft. above a datum, a 12-inch pipe runs 12,000 ft. to a second reservoir whose level is at elevation 220 ft. A valve midway along the line is closed sufficiently to reduce the discharge to one half what it was with the valve wide open. The friction factor, f, may be taken as 0.023 with valve open and as 0.021 when it is partly shut. Compute the loss of head due to the partial closure of the valve, and sketch the hydraulic grade line. The loss at pipe entrance may be neglected, also that consumed in giving the water its velocity.

Ans. 62.7 ft.
61.7

19. At a point A, a 12-inch pipe is 400 ft. above a given datum. It terminates 6000 ft. beyond A, and at elevation 500, in the bottom of a standpipe which contains 30 ft. of water. With flow toward the standpipe, the hydraulic grade line at A is 150 ft. above the pipe. What is the rate of discharge into the standpipe if f be 0.02?

Ans. 2.6 cu. ft. per second.

20. The slope of the hydraulic grade line of a pipe is 0.005 and the pipe delivers 4000 gallons per minute. If the friction coefficient be assumed as that for a clean pipe plus a 50% increase to allow for roughening of surface with age, find the diameter of the pipe.

Ans. 20 in.

21. What head would be required for an 8-inch wood-stave pipe line, 3000 ft. long, leading from a reservoir and terminating in a 2-inch nozzle, the required discharge being that corresponding to a velocity of flow of 6 ft. per second in the pipe? Assume velocity coefficient for the nozzle at 0.95 and $f=0.0223$.

Ans. 215 ft.

22. A 24-inch steel pipe line leaves a reservoir at elevation 1400 and runs 8000 ft. on a straight grade to elevation 1300; thence on a straight grade 4000 ft. to elevation 700, where it terminates in a 4-inch nozzle. The reservoir level being at elevation 1450, sketch the hydraulic grade line giving elevations at a sufficient number of points to define it. Assume $f=0.025$ and $c_v=c_d=0.95$ for the nozzle. At 80% efficiency, what horse power can be developed by a wheel driven by the jet?

Ans. 960 H. P.

23. A pipe line 30,000 ft. long and 6 ft. in diameter supplies 10 nozzles with water from a reservoir whose level is 507 ft. above the nozzles. Each nozzle has an opening of 6 sq. in. and a coefficient of discharge and velocity of 0.95. Assuming $f=0.017$, find the aggregate horse power available in the jets.

Ans. 3625 H. P.

24. A 24-inch pipe line, 5000 ft. long is to take water from a reservoir at El. 500 and discharge it through a 6-inch nozzle ($c_d=c_v=.95$) at El. 100. The pipe line is to be constructed so that at a point, A, 4500 ft. from the reservoir, the pipe shall be 10 ft. below the hydraulic grade line. What would be the elevation of the pipe (to nearest $\frac{1}{10}$ ft.) at A and what H. P. will be available in the jet from the nozzle? Assume $f=.025$ and no loss at entrance to pipe.

25. A pump at elevation 900 is pumping 1.60 cu. ft. per second through 6000 ft. of 6-inch pipe to a reservoir whose level is at elevation 1250.

What pressure will be found in the pipe at a point where the elevation is 1020 ft. above datum and the distance (measured along the pipe) from the pump 2500 ft.? Assume $f = .0225$.

Ans. 168 lb. per square inch.

26. From a reservoir whose surface is at elevation 750, water is pumped through 4000 ft. of 12-inch pipe across a valley to a second reservoir whose level is at elevation 800. If, during pumping, the pressure is 80 lb. per square inch at a point on the pipe, midway of its length and at elevation 650, compute the rate of discharge and the power exerted by the pumps. Plot the hydraulic grade line ($f = 0.02$). *Ans.* 5.8 cu. ft. per second. 78.5 H. P.

27. A fire-engine supplies water to a nozzle through 500 ft. of 3-inch hose. What power at the pump will be necessary to maintain a stream of water having a velocity of 75 ft. per second, with the nozzle held 30 ft. above the pump cylinder? The nozzle has a diameter of $1\frac{1}{2}$ in. at the tip and a coefficient of 0.90. The value of f for the hose may be assumed at 0.017.

Ans. 34 H. P.

28. Water is pumped from reservoir A, at Elev. 100, through 4000 ft. of 2 ft. pipe to reservoir B at Elev. 300. The pump is close to reservoir A and at the same elevation as A. A hydraulic motor is placed in the pipe line at a point midway on the length of pipe, and at Elev. 275. The motor delivers 56 H. P. and is 80% efficient. With a water delivery of 31.4 cu. ft. per sec. what will be the power output of the pump? What will be the pressures at entrance to, and exit from, the motor? Plot the grade line showing its elevations at controlling points. Assume $f = .02$.

29. A pump, taking water from a reservoir at Elev. 0, pumps it through a 12-inch diameter pipe over a hill whose summit is at Elev. 160 and is 2000 ft. from the pump. The pipe continues 1000 feet farther when it terminates in a 3-inch nozzle. The pump is to deliver 6 cu. ft. per sec. and the gauge pressure at the top of the hill is to be 60 lb. per sq. inch. Assume $f = .02$ for the pipe and $c_v = c_d = 0.95$ for the nozzle. Neglect entrance loss at the pump.

Compute, (a) the horse-power output of the pump,
(b) the height of the nozzle above the pump,
(c) the elevations of the hydraulic gradient at the pump, top of hill, base of nozzle and tip of nozzle.

30. Reservoir A is at elevation 1000 ft. above datum. Thence an 8-inch pipe line leads 3000 ft. to elevation 800, at which point it branches into

two lines: a 6-inch line running 2000 ft. to reservoir B, elevation 850, and a 6-inch line running 1000 ft. to reservoir C, elevation 875. At what rate will water be delivered to each reservoir? Assume $f = 0.02$ in all cases.

Ans. To B, 1.35 cu. ft. per second.
To C, 1.45 cu. ft. per second.

31. A reservoir at elevation 300 ft. above datum furnishes water to a 24-inch pipe which leads to a point at elevation 100 ft., the pipe being 2000 ft. long. Here it branches into 3 pipes, 8 inches, 12 inches, and 6 inches in diameter. The 8-inch runs 1000 ft. and discharges at elevation 250, the 12-inch runs 1500 ft. to elevation 175, and the 6-inch runs 3000 ft. and discharges at elevation 100. Compute the discharge for each pipe. Assume $f = 0.02$.

Ans. From 8-in., 3.24 cu. ft. per second.
From 12-in., 12.33 cu. ft. per second.
From 6-in., 1.98 cu. ft. per second.

32. Reservoir No. 1 is at Grade 400. Thence an 18-inch pipe, which is to carry 7 cubic feet per second of water, leads 2000 feet to Grade 300. It there divides and Branch A, 12 in. in diameter, leads 13,000 feet to Reservoir No. 2, which is at Grade 250. Branch B leads 4000 feet to Reservoir No. 3, which is at Grade 50. Neglect all losses except from friction, assume f as .02 in each case, and find the diameter of Branch B.

33. A 12-inch pipe, 8000 ft. long, is connected with a reservoir whose surface is 250 ft. above the pipe's discharging end. If for the last 4000 ft. a second pipe of the same diameter be laid beside the first and connected to it, what would be the increase in discharge? Assume $f = 0.02$.

Ans. 2.1 cu. ft. per second.

34. Assuming the pipe line as described in the first part of the preceding problem, find the change in discharge resulting from inserting in the original line a section of 18-inch pipe 2000 ft. long. (No restriction is made regarding the location of the enlarged portion.) Loss by change in section may be neglected and f assumed equal to 0.02.

Ans. 1.1 cu. ft. per second.

35. A 6-inch pipe leaves a straight 4-inch pipe at a point A and later joins it again at a point C. The distance AC on the straight 4-inch pipe is 2000 ft. and on the 6-inch pipe it is 15,000 ft. How will the flow divide when it comes to A? Assume $f = 0.02$ for both pipes and consider only losses by pipe friction.

Ans. Ratio 1 to 1.

36. A 48-inch main, carrying 75.4 sec. ft. branches at a point A into two pipes, one 2000 ft. long, 3 ft. in diameter, and one 6000 ft. long, 2 ft. in diameter. Both pipes come together at a point B and continue as a single 48-inch pipe. The following value of f may be assumed: $f = .0210$, 0.022 and 0.023 respectively for the 48-inch, 36-inch and 24-inch pipes. Compute the rate of flow in the branch pipes.

37. A six-inch pipe leaves a straight four-inch pipe at a point A, and later joins it again at C. The distance AC on the straight four-inch pipe is 2000 feet. How long will the six-inch pipe have to be in order that the flow in the two pipes may be the same? Assume $f = .02$ and consider only losses by pipe friction.

38. Water is pumped through 3000 ft. of 6-inch pipe to a reservoir whose level is 20 feet above the pump. During pumping, a gauge on the line at a point 40 feet above the pump and 1000 feet away from it (measured on the pipe) reads 20 lb. per sq. inch. At the same time a gauge on the 6-inch *suction* pipe, close to the pump, reads 7.35 lb. per sq. inch below atmosphere. Against what head is the pump working? What energy does the pump impart to the water per second? Assume $f = 0.02$.

39. A 6-inch pipe line, 2400 feet long, has its discharging end 20 feet below the level of the reservoir which supplies it. The discharge is 0.50 cu. ft. per second and $f = 0.025$. Compute the loss in head which is being caused by a partial obstruction that exists in the pipe.

40. Two open, cylindrical tanks are connected by 1000 ft. of 3-inch iron pipe laid horizontally. Reservoir A is 25 ft. in diameter and its water level is 36 ft. above that in reservoir B whose diameter is 16 ft. How long, after opening a valve on the pipe line, will it be before the reservoir levels are the same? Assume f to be constant at 0.02 and neglect head lost at entrance.

Ans. 10 hr. 54 min.

41. The tip of the Pitot tube being placed at the center of a 24-inch pipe, the difference in the height of the velocity and pressure columns was found to be 22.3 inches of water. On the assumption that the curve of velocities was an ellipse, and that the side or surface velocity was one half that at the center, compute the discharge from the pipe. The tube's coefficient, c, was 1.05 in the formula, $h = c\,\frac{v^2}{2g}$.

Ans. 27.8 cu. ft. per second.

42. A Cole pitometer (see Art. 114) was used in conjunction with a differential gauge to measure the velocity in a 10-inch pipe. Carbon

tetrachloride (spec. gr. 1.50) was used in the gauge and the deflection, z, noted as the pitometer tip was successively placed at 9 points on the pipe's diameter. Readings and corresponding distances from the pipe's axis follow:

Distance: 4.75″, 4.25″, 3.50″, 2.50″, 0.0″, 2.50″, 3.50″, 4.25″, and 4.75″.

Deflection z: 2.72″, 3.96″, 5.15″, 5.61″; 5.90″, 5.86″, 5.35″, 4.48.″, and 3.68″.

The coefficient of the tube (see equation 129) was 0.84.

(*a*) Show that the velocity corresponding to any reading was $v = 0.84\sqrt{2\,g\,(s—1)\,z}$.

(*b*) Make a scale plot showing variation in velocity across the pipe.

(*c*) Compute discharge. Assume pipe's section to be divided by concentric rings so spaced that each point occupied by the pitometer falls midway between rings. Each annular area is to be multiplied by the mean of the velocities observed in it to give partial discharge.

(*d*) Compute pipe's coefficient, *i. e.*, the ratio of mean velocity to center velocity.

43. A Cole pitometer is set with its orifices at the center of a 16-inch pipe. For three different rates of discharge the deflection, z, of the carbon tetrachloride (sp. gr. 1.50) in the differential gauge is 2, 7 and 13 inches. With $c = 0.84$, what are the three discharges? The ratio of the mean velocity to center velocity may be assumed as 0.83.

44. A Venturi meter has an area ratio of 9 to 1, the larger diameter being 12 in. During flow the recorded pressure-head in the large section is 21.4 ft. and that at the throat, 13.9 ft. If c be 0.99, what rate of discharge through the meter is indicated?

Ans. 1.91 cu. ft. per second.

45. A 24-inch Venturi meter has a throat diameter of 8 in. During a 10-minute test it discharged 18,600 cu. ft. of water with a mean pressure-head at the large section of 112 ft. and at the throat a negative pressure corresponding to 10.50 in. of mercury. Compute the coefficient of the meter.

Ans. 0.992.

46. The discharge throught a Venturi meter is 920 gallons per minute. The diameter of the pipe is 30 inches and the area ratio is 4 to 1. If the value of c be 0.99 and the pressure-head at entrance 18.8 feet, find the velocity and the pressure-head at the throat.

Ans. 18.76 ft.; 1.68 ft. per second.

47. A special Venturi meter has area ratios of 1 to 10. It is fitted to a 2-ft. pipe and the difference in pressure between the entrance and the throat corresponds to 18 inches of mercury. What discharge is indicated, assuming a coefficient of discharge of 0.98?

48. A velocity of 3 ft. per second is obtained through a 12-inch pipe whose hydraulic gradient slopes 3 ft. per 1000. Compute the value of the friction factor, f, and the amount of frictional resistance in pounds per square foot of pipe surface.

Ans. $f = 0.0215$; $F = 0.047$.

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CHAPTER IX

FLOW IN OPEN CHANNELS

118. Divisions of the Subject.—The term, *open channel*, is here used to include not only natural streams and all artificial canals, but also all forms of closed conduits which flow only *partly full*, or if filled, are so designed and built that the water in them exerts no pressure against their top elements. In the latter classification would fall a pipe, so designed with reference to slope and quantity of water carried, that its top element concided with the hydraulic grade line. A distinct difference between open channels and closed conduits under pressure is that the latter depend upon an external head for the production of flow, while the former depend upon the slope given their free surface or gradient.

The analytical treatment of open channel flow is more difficult and unsatisfactory than the flow in pipes, because of the wide range in the conditions which present themselves. A pipe is generally circular in form and the roughness of its lining is the chief variable. With an open channel not only is there found to be a wide variance in the nature of the lining, but also the form of cross-section may be of an almost infinite variety of shapes and may change from point to point even in the same channel. Under these circumstances it can be readily seen that it is exceedingly difficult to derive a formula for flow that shall be perfectly general in its application. Indeed it has so far been impossible to do so, and there is little or no likelihood of such a formula being discovered.

Broadly viewed, all problems of open channel flow may be classified under the following divisions:—

1. Flow in Artificial Channels.
   (*a*) Uniform Flow.
   (*b*) Non-uniform Flow.
2. Flow in Natural Channels.

119. Artificial Channels, Uniform Flow.—Here, as in all previous subjects treated, the condition of steady flow will be

assumed, so that the quantity of water passing any section of the stream is constant. To make the flow *uniform*, all cross-sections must be identical in form and area, necessitating *constant depth* and *constant mean velocity* at each section. Under these conditions the surface of the water is parallel to the bed, both having an angle of inclination, α, with the horizontal. The inclination of the surface we will speak of as the *slope* of the channel and express it as

$$S = \frac{h}{l}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132)$$

h being the vertical fall occurring in the length of channel, l, and S the *sine* of α (Fig. 112). In any cross-section $ABCD$ (Fig. 113), that part of the channel lining which comes in contact

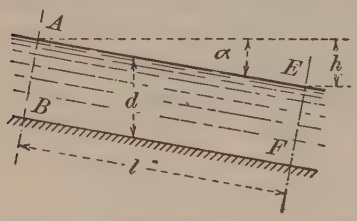


Fig. 112

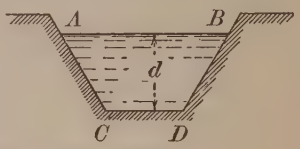


Fig. 113

with the stream is known as the *wetted perimeter*. In two channels of equal area, having like slopes and walls of like materials, it is obvious that the channel having the smaller wetted perimeter will have the higher velocity of flow on account of lesser frictional resistance. The ratio of the area to the wetted perimeter is therefore an important factor in the rate of flow and to it is given the name of *hydraulic radius* or *hydraulic mean depth*.

$$\text{Hydraulic Radius} = R = \frac{\text{Area}}{\text{Perimeter}}.$$

The term has little significance in itself. It should be regarded as a name for an oft-recurring ratio.

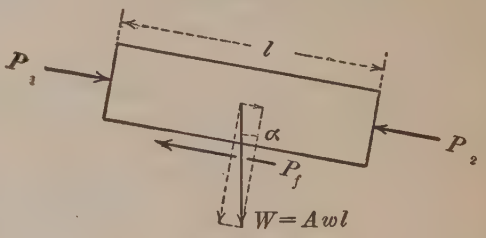


Fig. 114

Referring to Fig. 112, we may regard the water between any two sections, A-B and E-F, as a solid prism having a uniform motion down the inclined trough of the channel. The forces producing and hindering

motion are its weight Awl, the end pressures P_1 and P_2, and the frictional resistance P_f offered by the sides of the channel (Fig. 114). The forces P_1 and P_2 are equal since they represent the pressures on two equal areas under equal pressure-heads. Since they are opposite in direction, we need not consider them. The component of the prism's weight W, along the line of motion, being Awl sine α, and the motion being *uniform*, we have

$$Awl \text{ sine } \alpha = P_f.$$

If F represents the value of the frictional resistance per unit area of rubbing surface, so that

$$P_f = Fl \times \text{wetted perimeter} = Fl \times \frac{A}{R},$$

the above equation may be written

$$F = \frac{Awl \sin \alpha}{Al} \times R, \text{ or } F = w R S. \quad . \quad . \quad . \quad (133)$$

Our knowledge of fluid friction, as outlined in the previous chapter, shows F to vary approximately with v^2, the approximation being expressed in the exponent of v. If we assume

$$F = cv^2,$$

c being a constant whose value depends on the channel lining, equation (133) may be written,

$$v = C\sqrt{RS}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (134)$$

in which C has replaced $\sqrt{\frac{w}{c}}$. This formula is known as Chezy's formula, being identical with the one deduced in Art. 85 to express the law of flow in pipes. At first glance the two formulæ would seem to differ as, in the case of the pipe, S represents the slope of the hydraulic grade line, while in the open channel it represents the slope of the stream's surface. For the open channel the grade line lies in the stream's surface, since the latter would mark the heights of all open piezometer columns which might be placed in it. The two formulæ are, therefore, identical.

In the derivation just given all fairly rational treatment of the problem ceased with the assumption that $F = cv^2$. This relation we have seen to be only an approximate one. If it were strictly true, C in Chezy's formula would be constant for any particular

channel, varying only with the roughness of its lining. Experiments show that this is not the case, and that C varies also with the slope and the hydraulic radius; and that it may be made approximately a constant by giving to R and S fractional exponents which are more closely in agreement with facts than is the exponent one half. Thus has it been proposed to write

$$v = K R^{\alpha} S^{\beta},$$

and considerable work has been done by experimenters toward determining proper values for α and β. The chief difficulty lies

12-Foot Semi-circular Wood Flume.—Central Oregon Irrigation Project.
(Courtesy of Continental Pipe Mfg. Co.)

in the fact that α and β have been found to vary somewhat in channels having similar or like linings and that any values assigned them for general use must necessarily be average values.

The Williams and Hazen formula for pipe flow (see Art. 105) may be used for open channels and is one that has average values for α and β. The Schoder formulæ of Art. 108 illustrate the variance of α and β with roughness of lining.

The bulk of experimental work on channels has been directed

toward establishing the value of C in the Chezy formula and a brief outline of some of the well-known investigations will now be given.

120. Determination of C by Ganguillet and Kutter.—In 1869 Messrs. Ganguillet and Kutter made a most comprehensive study of all the reliable open-channel experiments which had been made up to that time, and from them deduced a formula for C which is the only general rule of wide application that we have. The experiments from which they drew their conclusions ranged from observations on small artificial channels up to measurements made on the Mississippi River. Their formula is as follows:—

$$C = \frac{41.65 + \frac{0.00281}{S} + \frac{1.811}{n}}{1 + \left(41.65 + \frac{0.00281}{S}\right)\frac{n}{\sqrt{R}}} \quad . \quad . \quad . \quad (135)$$

It is seen that the value of C is made to vary with the slope, the hydraulic radius, and a quantity n which may be termed the *coefficient of roughness*. The latter depends upon the channel lining and has values approximately as follows:—

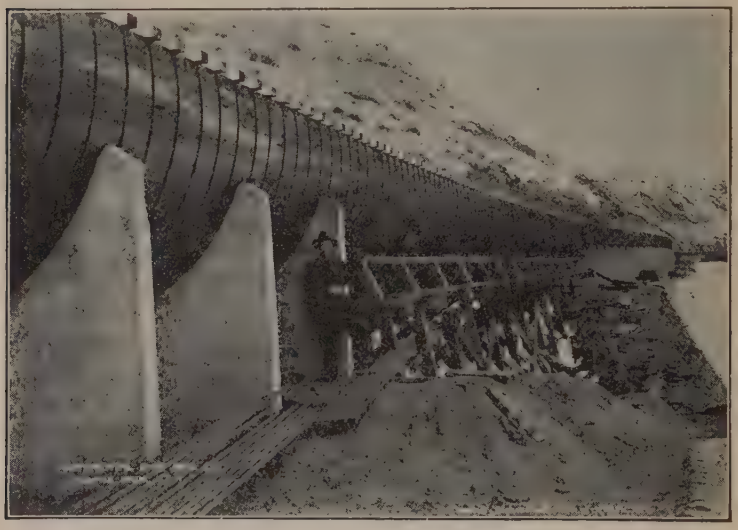
CHANNEL LINING

	n
Planed timber, glazed or enameled surfaces	0.009
Smooth clean cement	.010
Unplaned timber, new brickwork well laid	.012
Smooth stonework, iron, and ordinary brickwork	.013
Rough ashlar and good rubble masonry	.017
Firm gravel	.020
Earth in ordinary condition	.025
Earth with stones, weeds, etc	.030
Earth or gravel in bad condition, strewn with detritus	.035

Kutter's formula has been widely used by American, German, and English engineers and may be relied upon to give excellent results when applied to normal channels. By a normal channel is meant one whose slope, hydraulic radius, or velocity of flow has no unusual or extreme value. A great deal has been claimed for the formula, since it was designed to apply to not only regular channels of uniform section, but also to natural rivers. Inasmuch as the basis of the formula is the assumption of steady, *uniform flow*, it would hardly seem probable that it can be depended upon

to give accurate results if applied to a channel whose general shape, slope, or hydraulic radius is constantly changing. The most that can be said for it when so used is that it may serve to roughly approximate the flow, and give a result which is liable to be much in error. It is quite safe to say that any formula which is derived for steady, uniform flow in regular, artificial channels, is hardly suitable for use with large, irregularly shaped, natural streams.

The complication of the formula, and the inclusion of S, may be said to be largely due to the effort of its authors to make it conform to the results of the measurements made on the Mississippi



137-Inch Wooden Flume on the Columbia River.
(Courtesy of Continental Pipe Mfg. Co.)

River. Although these were made with due care, it is probable that errors made in measuring the very small slopes and the mean velocities, render the results unreliable. It is doubtful, in view of recent experiments, that the value of C depends upon the hydraulic slope, S, and other investigators have generally concluded that the effect of S was either very slight or negligible. The wide acceptance of Kutter's formula has given it considerable authority and its complicated form is no deterrent from its use as tables and diagrams have been prepared to give values of C for all common values of n, R and S. Such a table appears opposite.

COEFFICIENT (C) IN KUTTER'S FORMULA

Slope S	n	HYDRAULIC RADIUS R IN FEET														
		0.2	0.3	0.4	0.6	0.8	1.0	1.5	2.0	2.5	3.0	4.0	6.0	8.0	10.0	15.0
.00005	.010	87	98	109	123	133	140	154	164	172	177	187	199	207	213	220
	.012	68	78	88	98	107	113	126	135	142	148	157	168	176	182	189
	.015	52	58	66	76	83	89	99	107	113	118	126	138	145	150	159
	.017	43	50	57	65	72	77	86	93	98	103	112	122	129	134	142
	.020	35	41	45	53	59	64	72	80	84	88	95	105	111	116	125
	.025	26	30	35	41	45	49	57	62	66	70	78	85	92	96	104
	.030	22	25	28	33	37	40	47	51	55	58	65	74	78	83	90
.0001	.010	98	108	118	131	140	147	158	167	173	178	186	196	202	206	212
	.012	76	86	95	105	113	119	130	138	144	148	155	165	170	174	180
	.015	57	64	72	81	88	93	103	109	114	118	125	134	140	143	150
	.017	48	55	62	70	75	80	88	95	99	104	111	118	125	128	135
	.020	38	45	50	57	63	67	75	81	85	88	95	102	107	111	118
	.025	28	34	38	43	48	51	59	64	67	70	77	84	89	93	98
	.030	23	27	30	35	39	42	48	52	55	59	64	72	75	80	85
.0002	.010	105	115	125	137	145	150	162	169	174	178	185	193	198	202	206
	.012	83	92	100	110	117	123	133	139	144	148	154	162	167	170	175
	.015	61	69	76	84	91	96	105	110	114	118	124	132	137	140	145
	.017	52	59	65	73	78	83	90	97	100	104	110	117	122	125	130
	.020	42	48	53	60	65	68	76	82	85	88	94	100	105	108	113
	.025	30	35	40	45	50	54	60	65	68	70	76	83	86	90	95
	.030	25	28	32	37	40	43	49	53	56	59	63	69	74	77	82
.0004	.010	110	121	128	140	148	153	164	171	174	178	184	192	197	198	203
	.012	87	95	103	113	120	125	134	141	145	149	153	161	165	168	172
	.015	64	73	78	87	93	98	106	112	115	118	123	130	134	137	142
	.017	54	62	68	75	80	84	92	98	101	104	110	116	120	123	128
	.020	43	50	55	61	67	70	77	83	86	88	94	99	104	106	110
	.025	32	37	42	47	51	55	60	65	68	70	75	82	85	88	92
	.030	26	30	33	38	41	44	50	54	57	59	63	68	73	75	80
.001	010	113	124	132	143	150	155	165	172	175	178	184	190	195	197	201
	012	88	97	105	115	121	127	135	142	145	149	154	160	164	167	171
	015	66	75	80	88	94	98	107	112	116	119	123	130	133	135	141
	017	55	63	68	76	81	85	92	98	102	105	110	115	119	122	127
	.020	45	51	56	62	68	71	78	84	87	89	93	98	103	105	109
	.025	33	38	43	48	52	55	61	65	68	70	75	81	84	87	91
	.030	27	30	34	38	42	45	50	54	57	59	63	68	72	74	78
.01	.010	114	125	133	143	151	156	165	172	175	178	184	190	194	196	200
	.012	89	99	106	116	122	128	136	142	145	149	154	159	163	166	170
	.015	67	76	81	89	95	99	107	113	116	119	123	129	133	135	140
	.017	56	64	69	77	82	86	93	99	103	105	109	115	118	121	126
	.020	46	52	57	63	68	72	78	84	87	89	93	98	102	105	108
	.025	34	39	44	49	52	56	62	65	68	70	75	80	83	86	90
	.030	27	31	35	39	43	45	51	55	58	59	63	67	71	73	77

The effect of S on the value of C may be seen from an inspection of the table. Between slopes of 0.01 and 0.001 the variation in C is small and, for slopes steeper than 0.001 the value of C computed for $S = 0.001$ may be used without incurring errors larger than are inherent to the formula. For $S = 0.001$ the value of C becomes

$$C = \frac{44.4 + \dfrac{1.811}{n}}{1 + 44.4\dfrac{n}{\sqrt{R}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (136)$$

121. Determination of C by Bazin.—In 1897, H. Bazin (Annales des Ponts et Chaussées, 1897) made a most elaborate and careful discussion of his own, and all other reliable experiments, and proposed a formula for C which may, for English units of measure, be written as follows,—

$$C = \frac{157.6}{1 + \frac{m}{\sqrt{R}}}. \quad \ldots\ldots\ldots \quad (137)$$

The quantity R is the hydraulic radius and m is a coefficient of roughness like the n in Kutter's formula. Values for m may be taken as follows:—

CHANNEL LINING

	m
For very smooth cement surfaces or planed boards	0.109
For unplaned boards, well laid brick, or concrete	0.290
For ashlar, good rubble masonry, or poor brickwork	0.833
For earth beds in perfect condition	1.54
For earth beds in fair or ordinary condition	2.35
For earth beds in bad condition, being covered with sticks, stones, weeds, and other detritus	3.17

Values of C computed from (137) for various values of R and m are given in the following table:—

COEFFICIENTS (C) FROM BAZIN'S FORMULA

HYDRAULIC RADIUS R IN FEET	COEFFICIENT OF ROUGHNESS m					
	m = .109	m = .29	m = .833	m = 1.54	m = 2.35	m = 3.17
0.2	126	96	55	36	25	19
.3	132	103	63	41	30	23
.4	134	108	68	46	33	26
.5	136	112	71	50	36	29
.75	140	118	80	57	42	34
1.0	142	122	86	62	47	38
1.25	143	125	90	66	51	41
1.50	145	127	94	70	54	44
2.0	146	131	99	76	59	49
2.5	147	133	104	80	63	53
3.0	148	135	106	83	67	57
5.0	150	140	115	93	77	65
7.0	152	142	120	100	83	72
10.0	152	144	125	106	91	79
12.0	153	145	127	109	94	82
16.0	153	147	130	114	99	88
20.0	154	148	133	117	103	92

It will be noted that Bazin's formula makes C independent of the slope, its author believing that the latter had little influence on the coefficient. The formula is much simpler than Kutter's and has been extensively used in France. In general it will be found to give smaller results than Kutter's formula, and its use should be restricted to small artificial channels since it was derived from experimental data largely obtained from such channels.

122. Determination of C by Manning.—In 1890 Manning, an Irish engineer, suggested the following formula for open channels:—

$$v = \frac{1.486}{n} R^{\frac{2}{3}} S^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (138)$$

This may be written

$$v = \frac{1.486}{n} R^{\frac{1}{6}} \sqrt{R\,S}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (139)$$

which is the Chezy formula with

$$C = \frac{1.486\ R^{\frac{1}{6}}}{n}.$$

The quantity n is a coefficient of roughness and identical in value with the n of Kutter. It will be noted that this expression for C is even more simple than the Bazin and, like the latter, makes C independent of S.

The following table has been constructed to show values of C for various values of n and R. It should be studied with reference to the one prepared from Kutter's formula and the degree of agreement noted.

COEFFICIENT (C) FROM MANNING'S FORMULA

	HYDRAULIC RADIUS R IN FEET															
n	0.2	0.3	0.4	0.6	0.8	1.0	1.5	2.0	2.5	3.0	4.0	6.0	8.0	10	15	20
.010	112	120	126	136	142	148	159	167	173	178	187	200	210	218	234	246
.012	95	102	106	114	120	124	133	139	144	149	156	166	175	182	194	204
.015	76	81	85	91	96	99	106	111	115	119	125	133	140	145	155	163
.017	67	71	75	80	84	87	93	97	100	105	109	116	122	126	135	142
.020	57	61	64	68	72	74	80	84	86	89	93	100	104	108	116	122
.025	45	48	51	54	57	59	63	66	68	70	74	79	83	86	92	97
.030	38	40	42	45	48	50	53	55	57	59	62	66	69	72	77	81
.035	32	35	36	39	41	43	46	48	49	51	53	57	60	62	66	69

123. Comparison of the Kutter and Manning Formulas.—The constants of both formulas were originally expressed in metric units. If the value of R in each formula be assumed at 3.28 feet (1 meter), it will be found that

$$c = \frac{1.811}{n},$$

or the two formulas give identical results for this value of R. The above equation does not contain S, hence we may reason that

High Level Canal on the Platte River, Colo.

the Kutter formula will give results unaffected by values of S, so long as $R = 3.28$ feet. This is a weakness of the formula, as

it would be absurd to argue that slope could effect the value of C in every channel save one having a hydraulic radius of 3.28 feet. The formula is entirely empirical, however, and such an absurdity is not surprising.

As to agreement between the Manning and Kutter formulas for values of R other than 3.28 feet, it may be stated that,—

For $R < 3.28$ feet, Kutter's gives generally higher values of C than does Manning's.

For $R > 3.28$ feet, Kutter's results are sometimes larger, sometimes smaller than Manning's.

In general, results by both formulas will be found to agree remarkably well except in cases of flat slopes having values less than 0.0004. In view of the above facts it seems that the Manning formula offers all the accuracy of Kutter's and possesses none of its cumbersomeness. It may be used in either its original form (138) or as altered in (139).

Another advantage found in the use of Manning's formula is that a very simple relation exists between any given value of n and corresponding values of v or S. An inspection of the formula shows that v varies inversely as n, and S varies as n^2, for all values of R. If we double n, other things remaining constant, the value of v will be halved; if we wish to maintain v constant with the new n, S must be made four times greater than its first value. The importance of these facts may be recognized by considering the problems which continually confront the designing engineer. His greatest difficulty lies in properly estimating the value of the roughness coefficient, n, and it becomes of prime importance that he know what error in computed values of v or S results from an error in the selection of n. From the relations stated above, this question can be at once answered. If a certain error be made in selecting n, then a computed value of v (or Q) will contain the same percentage error but in the opposite direction. Likewise a value of S computed to give a certain velocity will contain double the same percentage error.

Schoder has pointed out that the same relations hold approximately for the Kutter formula. "Other things being equal, the slope S varies as n^2 (almost exactly for all values of R greater than 1 foot); the velocity v varies inversely as n, exactly for R = about 2 feet, and approximately for other values."

No such simple relation exists between n and its dependents, v and s, in Bazin's formula.

124. The Kutter and Bazin Formulas.—It is not generally understand that in some respects the Bazin formula closely resembles Kutter's. If a moderate slope having a value of 0.00281 be chosen, the fraction, $\frac{0.00281}{S}$, in Kutter's formula may be replaced by unity. If a medium value of 0.0157 be used for n, the fraction $\frac{1.811}{n}$ becomes 115. For these values Kutter's formula reduces to

$$C = \frac{158}{1 + \frac{4.26\,n}{\sqrt{R}}},$$

which is identical with the Bazin formula if Bazin's m be replaced by $42.6\,n$. The chief difference in the two formulas lies in the fact that m does not always equal $42.6\,n$, but varies from 11 to 90 n. Schoder first called attention to this fact in "The Engineering News" for August, 1912.

Example.—Compute the rate of flow in a circular, brick-lined conduit, 5 ft. in diameter, having a slope of 1 in 1000. Conduit flows half full.

$$r = \frac{A}{p} = \frac{D}{4} = 1.25 \text{ ft.} \quad S = 0.001$$

By Kutter, $n = .012$ and C (table, p. 233) $= 131$

$$v = 131\sqrt{\frac{1.25}{1000}} = 4.65 \text{ ft. per sec.}$$

$$Q = 9.82 \times 4.65 = 45.6 \text{ cu. ft. per sec.}$$

By Manning, $n = 0.012$ and C (table, p. 235) $= 129$

$$v = 129\sqrt{\frac{1.25}{1000}} = 4.57 \text{ ft. per sec.}$$

$$Q = 9.82 \times 4.57 = 44.9 \text{ cu. ft. per sec.}$$

By Bazin, $m = 0.29$ and C (table, p. 234) $= 125$,

$$v = 125\sqrt{\frac{1.25}{1000}} = 4.43 \text{ ft. per sec.}$$

$$Q = 9.82 \times 4.43 = 43.5 \text{ ft. per sec.}$$

The results by Kutter and Manning are practically identical. Bazin's coefficient gives a slightly smaller value.

125. Most Advantageous Cross-section.—If an open channel has its slope, S, and cross-sectional area, A, *fixed*, it is evident that the maximum velocity (hence maximum discharge) will occur when the area is so shaped and proportioned that the wetted perimeter, or the surface of contact between the water and the channel lining, will be as small as possible (frictional resistance being reduced to a minimum). Since $R = \frac{\text{Area}}{\text{Wetted Perimeter}}$, the same result is achieved by making R a maximum. If the Chezy formula be solved for S, we have

$$S = \frac{v^2}{C^2 R},$$

and it is apparent that a given velocity (hence discharge) may be maintained at a minimum slope by making R a maximum. Evidently the channel of greatest *hydraulic* efficiency results from making R a maximum, and were nothing else to be considered all channels should be designed on this basis. Practical considerations, however, often make it inadvisable to follow this as a rule, but as a fact it should always be kept in mind and used as a guide. It becomes important, therefore, to determine what shapes and proportions result in maximum values of R.

Since of all figures having equal areas the circle has the least perimeter, it follows that an open channel will have the smallest possible wetted perimeter (hence maximum R) if the given area

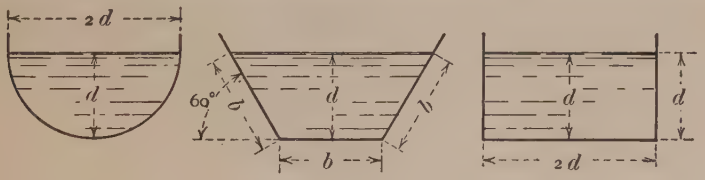


Fig. 115

be inclosed by a semi-circle (Fig. 115*a*). Such sections are common in small sizes but necessitate the use of a lining. More often a channel is built with a rectangular or trapezoidal section. Since

of all rectangles having the same area the square has the least perimeter, if follows that a channel of rectangular section should have the shape and proportions of a half-square in order to have a maximum value for its hydraulic radius (Fig. 115*c*). Similarly, if the channel section is to be trapezoidal (Fig. 115*b*), the minimum perimeter will be obtained if the section has the proportions of a half hexagon. Furthermore it should be noted that if the three channels illustrated all have the same area, the hydraulic radius decreases in value as we pass from the semi-circle to the trapezoid and from the trapezoid to the rectangle. The trapezoidal section shown has a larger R than that of the rectangle because its perimeter approaches more nearly the shape of a semi-circle. For all three sections the value of R will be found to be $\frac{d}{2}$ or half the depth.

In unlined channels it is necessary to use the trapezoidal section if the earth banks are to maintain their form. Generally, the sides are sloped at equal angles, but it often happens that one side is made vertical, and lined, while the other is sloped. As a general case, therefore, it will be assumed that both sides are sloped but at different angles, α and β (Fig. 116). The latter may have any values. In this case we may proceed as follows to determine the proportions which, for a given area, A, make R a maximum.

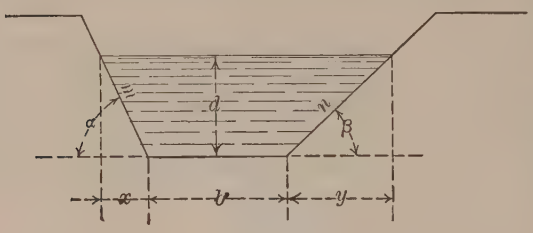


FIG. 116

From the figure,

$$\frac{x}{d} = \cot \alpha = a$$

$$\frac{y}{d} = \cot \beta = c$$

$$\frac{d}{m} = \sin \alpha = e$$

$$\frac{d}{n} = \sin \beta = f$$

all constants, so that

$$x = da, \qquad y = dc, \qquad m = \frac{d}{e} \text{ and } n = \frac{d}{f}.$$

For the area, we have

$$A = bd + \frac{xd}{2} + \frac{yd}{2} = bd + \frac{d^2}{2}(a + c),$$

and

$$b = \frac{A - \frac{d^2}{2}(a + c)}{d}.$$

$$\text{Wetted Perimeter} = m + n + b = \frac{d}{e} + \frac{d}{f} + \frac{A - \frac{d^2}{2}(a + c)}{d}$$

$$= \frac{d^2\left(\frac{1}{e} + \frac{1}{f}\right) + A - \frac{d^2}{2}(a + c)}{d}$$

$$= \frac{A + d^2\left(\frac{1}{e} + \frac{1}{f} - \frac{a + c}{2}\right)}{d}.$$

$$\text{Hydraulic Radius, } R, = \frac{Ad}{A + d^2\left(\frac{1}{e} + \frac{1}{f} - \frac{a + c}{2}\right)}, \text{ in which}$$

d is the only independent variable. If the first derivative of R with respect to d be obtained and placed equal to zero, the solution of the resulting equation will give that value of d which will make R a maximum.

$$\frac{\delta R}{\delta d} = \frac{A\left[A + d^2\left(\frac{1}{e} + \frac{1}{f} - \frac{a+c}{2}\right)\right] - Ad\left[2d\left(\frac{1}{e} + \frac{1}{f} - \frac{a+c}{2}\right)\right]}{\left[A + d^2\left(\frac{1}{e} + \frac{1}{f} - \frac{a + c}{2}\right)\right]^2} = 0,$$

or

$$A = d^2\left(\frac{1}{e} + \frac{1}{f} - \frac{a + c}{2}\right).$$

Substituting for A its value as given above, and solving for d, we obtain

$$d = \frac{b}{\frac{1}{e} + \frac{1}{f} - (a + c)} = \frac{b}{\csc \alpha + \csc \beta - (\cot \alpha + \cot \beta)}. \quad (140)$$

The corresponding value of R is

$$R = \frac{d}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (141)$$

If $\alpha = \beta = 90°$,

$$d = \frac{b}{2},$$

which proves the statement made earlier regarding the proportions of the most advantageous rectangular section.

Summarizing the foregoing proofs and statements, we may say that in semi-circular channels, and *all* trapezoidal channels, the best proportions will be such that the hydraulic radius will always equal the half-depth.

Example.—Determine the best proportions for a trapezoidal channel having an area of 100 sq. ft. and sides sloping at 45 degrees. *Solution.*

$$A = bd + d^2 = 100$$

$$b = \frac{100 - d^2}{d}$$

$$R = \frac{100}{b + 2d\sqrt{2}}$$

$$= \frac{100d}{100 - d^2(1 - 2\sqrt{2})}$$

Also $R = \frac{d}{2}$, and solving for d

$$d = 7.4 \text{ ft.}$$
$$b = 6.1 \text{ ft.}$$
$$R = 3.7 \text{ ft.}$$

This solution depends only upon a knowledge that R must equal $\frac{d}{2}$ and is therefore recommended to the student. Equation (140) may be used in combination with an equation for the area as follows:—

From (140),

$$d = \frac{b}{0.828}$$

$$A = 100 = bd + d^2$$

or
$$100 = 0.828\,d^2 + d^2$$
$$d = 7.4 \text{ ft.}$$

Example.—A trapezoidal section has one side vertical and the other sloping at 30 degrees from the horizontal. Determine the best proportions for an area of 300 sq. ft.

Solution. This time using equation (140),

$$d = \frac{b}{4 - 2\sqrt{3}}.$$

$$A = 300 = bd + \frac{2d^2}{\sqrt{3}}$$

or
$$300 = 0.54\,d^2 + \frac{2\,d^2}{\sqrt{3}}.$$

$$d = 13.3 \text{ ft.}$$
$$b = 7.2 \text{ ft.}$$
$$R = 6.7 \text{ ft.}$$

Example.—A triangular channel with equally sloping sides is to have an area of A sq. ft. Determine the vertex angle.

Solution. Denoting the surface width by w, the depth at the center by d, and the vertex angle by 2α,

$$\tan\alpha = \frac{w}{2\,d} = x, \text{ a variable,}$$

and
$$w = 2\,dx.$$

$$A = \frac{wd}{2} = d^2x$$

from which
$$d = \frac{A^{\frac{1}{2}}}{x^{\frac{1}{2}}}.$$

$$R = \frac{A}{2\sqrt{\frac{w^2}{4} + d^2}} = \frac{A^{\frac{1}{2}}x^{\frac{1}{2}}}{2\,(1 + x^2)^{\frac{1}{2}}}.$$

$$\frac{\delta R}{\delta x} = \frac{A^{\frac{1}{2}}(1 - x^2)}{4\,x^{\frac{1}{2}}\,(1 + x^2)^{\frac{3}{2}}} = 0$$

or
$$x = 1.$$
$$\alpha = 45°.$$

The vertex angle is therefore 90° and the center depth is one half the surface width.

This result might have been arrived at by noting that an isosceles triangle is one half a rhombus, and that of all rhombuses having equal area, the one having angles of 90° will have the least perimeter. It follows that a half square (divided on a diagonal) will produce a triangular channel having the best proportions.

126. Irregular Sections.—Fig. 117 represents a possible section which is quite irregular in that the depth suddenly changes by a considerable amount. Assuming $S = 0.0001$ and that the lining indicates $n = 0.02$, let us compute the rate of discharge without reference to the irregularity.

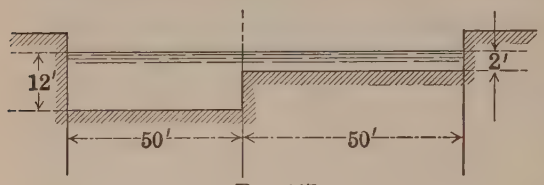


FIG. 117

From the dimensions given, the total area is found to equal 700 sq. ft. and the perimeter 124 ft.

$$R = 700 \div 124 = 5.65 \text{ ft.}$$

By Manning,

$$Q = 700 \times 99\sqrt{\frac{5.65}{10000}} = 1650 \text{ sec. ft.}$$

We will now divide the area into two sections as shown by the dotted line and compute the approximate discharge through each section. For the left portion,

$$A = 600$$

$$\text{Perimeter} = 74 \text{ (including length of dotted line).}$$

$$R = 600 \div 74 = 8.11$$

$$Q = 600 \times 104\sqrt{\frac{8.11}{10000}} = 1780 \text{ sec. ft.}$$

For the right portion,

$$A = 100$$

$$\text{Perimeter} = 54$$

$$R = 1.85$$

$$Q = 100 \times 83\sqrt{\frac{1.85}{10000}} = 112 \text{ sec. ft.}$$

Adding the partial discharges, $Q = 1892$ sec. ft., which is far in excess of the amount as first computed. In fact it should be noted that the discharge through the left hand portion alone figures more than as first computed for the whole section. The second computation resulting in 1892 sec. ft., is doubtless more correct than the first, but it must be regarded as only approximately representing the discharge.

Such sections are not common to artificial channels but natural streams may, in time of flood flow, offer similar conditions by reason of the banks being overflowed. While the author wishes to discourage the practice of ascertaining natural stream flow by the application of the Chezy or any other formula, it is here pointed out that when so doing, care must be taken to follow the above method if the section of the stream be very irregular with respect to depth.

127. Solution of Problems in Uniform Flow.—In the numerical solution of problems in channel flow, there are generally four factors involved,— v, R, S, and n (or m). If three of these are numerically known, the fourth may be directly obtained by the use of the Chezy formula, $v = C\sqrt{RS}$, using either the Manning or Bazin value for C. If the value of S be the unknown, the use of the Kutter C requires a tentative assumption of its value, since Kutter made C depend upon S. This tentative value may be estimated by noting the range in C, for the given value of R, as set forth in the table on p. 233. An approximate value for S being obtained, a close value of C is possible and a final solution for S made.

When two of the four quantities, v, R, S and n, are unknown, it will be found that the most direct method of solution consists in tentatively assuming the value of one, and proceeding by a method of trial. One of the unknowns will be R, and its value may be more easily estimated than that of the other unknown, since

it increases approximately with the size of the channel. Values of R and the corresponding *surface* width of the channel are given below for a semi-circular section, the best rectangular section, and a trapezoidal section having best proportions and side slopes of 30° with the horizontal.

	Semi-circular and Rectangular					Trapezoidal $\alpha = \beta = 30$ degrees			
R	1	2	3	5	6	1	2	3	5
w	4	8	12	20	24	8	16	24	40

The table shows that in channels of moderate size, R will usually have values less than 5 or 6 feet.

Example.—Determine the minimum hydraulic slope, S, for a rectangular channel that is to carry 600 sec. ft. with a mean velocity of 3 ft. per second. The lining is smooth concrete such that n in the Kutter and Manning categories of roughness may be assumed as 0.013.

Solution. To make S a minimum requires R to be a maximum, hence $b = 2d$ and $R = \frac{d}{2}$.

$$A = 200 = 2d^2$$

$$d = 10 \text{ and } r = 5.$$

From the table of Manning coefficients, $C = 150$.

$$3 = 150\sqrt{5S}$$

$$S = 0.00018 \text{ or } 0.18 \text{ ft. per } 1000.$$

Example.—A rectangular channel, 18 feet wide, is to carry 216 sec. ft. on a slope of 1 per 10000. If the lining be such that $n = 0.015$, determine the depth.

Solution. Here R and v are unknown. Were the section to have best proportions, the depth would be 9 ft. and R would be 4.5 ft. Since this is improbable, we may begin by assuming $R = 3$ ft. Using Manning's Coefficient, C will be 119 and

$$v = 119\sqrt{\frac{3}{10000}} = 2.1 \text{ ft.}$$

For $R = 3$,

$$3 = 18\,d \div (18 + 2\,d)$$

$$d = 4.5 \text{ ft.}$$

$$A = 81 \text{ sq. ft.}$$

$$Q = 81 \times 2.1 = 170 \text{ sec. ft. (too small).}$$

If R be assumed 3.5 ft.,—

$$C = 122$$

$$v = 2.3 \text{ ft.}$$

$$d = 5.7 \text{ ft.}$$

$$A = 102.6 \text{ sq. ft.}$$

$$Q = 236 \text{ sec. ft.}$$

This quantity is only slightly in excess of that demanded and the final velocity may be estimated at 2.2 ft. per sec., requiring an area of 98 sq. ft., or a reduction in depth from the last figure, of about 0.25 ft. This would give $d = 5.45$ ft., and if we say that $d = 5.5$ ft., the precision is all that the data warrants.

128. Other Formulas for Open Channel Flow.—It was pointed out in Art. 119 that the value of C for a particular channel would be independent of R and S if to these latter quantities exponents were given that truly represented the law of flow in that channel. Were this done, the formula for velocity would become

$$v = KR^{\alpha} S^{\beta},$$

and it would be necessary to know only the values of K, α and β that were proper for each channel lining. Experimental work has indicated that only average values for these constants are possible inasmuch as channel linings of like material appear to vary in their roughness and hence in their resistance to flow. The most notable attempt to catalogue values for the constants is that made by A. A. Barnes and discussed in the chapter on pipe flow. It is possible that further work may enable us to set up fair values for different linings which may be depended upon to give good results.

Williams and Hazen claim that their formula,

$$v = C\,R^{0.63}\,S^{0.54}\,0.001^{-.04},$$

which was quoted in Art. 105, may be used with good results for open channels. This may be written

$$v = 1.318\,C\,R^{0.63}\,S^{0.54},$$

the exponents of R and S being average values for all linings, and C varying moderately with the lining. Values of C are found in Art. 105.

129. Critical Velocity in Open Channels.—So far in discussing channel flow, it has been assumed that frictional resistance is proportional to the square of the velocity, or to a power approximating the square. As in the case of pipe flow, it is probable that at low velocities the resistance is proportional to the velocity, especially in smooth lined channels. Little data are available from which to formulate an expression for the value of the critical velocity, but what do exist, indicate higher values than would be found in pipes having a like hydraulic radius. Gibson remarks that the existence of a critical velocity would go far toward explaining discrepancies found among results of experiments upon channels of similar physical characteristics but having different velocities of flow.

130. Non-uniform Flow.—We have seen that in order to have *uniform* flow, a constant area and form of section must be maintained, and the mean velocity at all sections must have the same value. The water depth must also be constant.

Now it may happen that, owing to the presence of peculiar conditions, all these characteristics of uniform flow will be found absent. The flow will be steady, giving for all sections the relation,

$$Q = a_1v_1 = a_2v_2, \text{ etc.},$$

but the form and area of the sections will constantly change and the mean velocity will vary from section to section.

To illustrate, let Fig. 118 represent a channel constructed to connect two large ponds whose surfaces stand at different levels. The bed of the channel is *horizontal* and we will assume the pond levels to remain constant. Flow will take place in some such manner as indicated, the depth at A-B being in excess of that at

C-D. The cross-section and the velocity of the stream is, therefore, constantly changing between these points. The flow is steady but

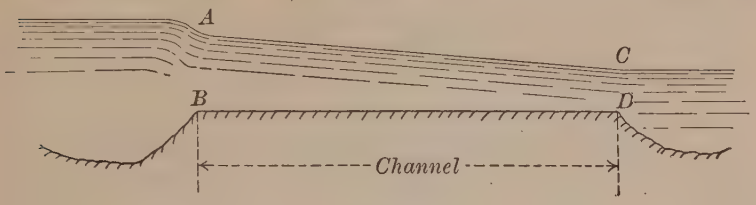


FIG. 118

non-uniform, and it cannot be made uniform until, by adjusting the slope to the flow, the bed of the stream and the water surface are *parallel*. In the figure given, the bed was shown horizontal, but it might have an upward slope from *B* to *D*, and the flow be maintained by the hydraulic slope of the surface. Similarly the bed slope might have been at a less or greater inclination than the slope of the surface. Such conditions render it very difficult to formulate the relations existing between the various elements of flow because the mean velocity and the hydraulic radius are continually changing. Formulas constructed to fit the conditions can be regarded as only approximately correct, and results obtained from them must not be given too much credence.

Figure 119 shows a portion of the channel as constructed with a bed slope less than that of the surface. It will be assumed that v_1 is the velocity in the surface at the section *A-B* and its numerical

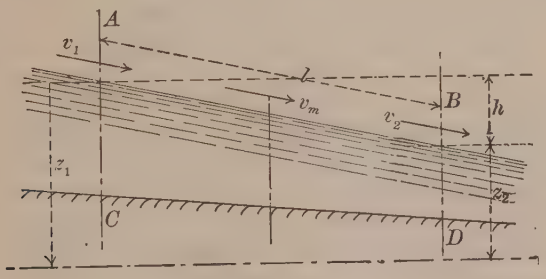


FIG. 119

value that of the mean velocity in the section. Similarly v_2 will be the value at section *B-D*. Between these two points in the surface we may write

$$\frac{v_1^2}{2g} + z_1 = \frac{v_2^2}{2g} + z_2 + \text{Lost head},$$

and since

$$z_1 - z_2 = h,$$

we have

$$\text{Lost Head} = h + \frac{v_1^2}{2g} - \frac{v_2^2}{2g}. \quad . \quad . \quad . \quad . \quad . \quad (142)$$

To otherwise express the lost head let us return for the moment to the case of uniform flow. By simple transposition of the Chezy formula, we obtain

$$s = \frac{v^2}{C^2 R}$$

or, since

$$s = \frac{h}{l},$$

$$h = \frac{lv^2}{C^2 R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (143)$$

Here h is the drop of the water surface or hydraulic grade line in any length of stream l. It therefore is the measure of the lost head in this distance. (The student should review Art. 119 in this connection and verify this statement.)

Returning now to the problem in non-uniform flow, it is seen that in the length l we have a constantly changing value for v and R; but we will *assume* that the head lost in this case can be expressed in the same form as (143), v and R having values which are average values for the length l. This value for the lost head being substituted in (142), there results after transposition,

$$\frac{v_1^2}{2g} + h = \frac{v_2^2}{2g} + \frac{l v_m^{\ 2}}{C^2 R_m}. \quad . \quad . \quad . \quad . \quad . \quad (144)$$

which may be regarded as the general equation for non-uniform flow. Just what values shall be given v_m and R_m in order to obtain best results is difficult to state. It is common practice to either compute them as means of the velocities and radii at the end sections, or to give them values derived from a consideration of the area of a mid-section. There is a noticeable difference in the results of the two methods. If the second be followed, and the assumption made that

$$A_m = \frac{A_1 + A_2}{2},$$

then it can be easily shown that

$$v_m = \frac{2v_1v_2}{v_1 + v_2}.$$

For the value of R_m, if we assume the wetted perimeter p_m to be equal to $\frac{1}{2}(p_1 + p_2)$, which we may do without sensible error, we have

$$R_m = \frac{A_1 + A_2}{p_1 + p_2}.$$

Also we may write

$$v_1 = \frac{Q}{A_1} \text{ and } v_2 = \frac{Q}{A_2}.$$

The substitution of the above values in equation (144) will give

$$Q = \frac{\sqrt{2gh}}{\sqrt{\left(\frac{1}{A_2}\right)^2 - \left(\frac{1}{A_1}\right)^2 + \frac{8gl}{C^2} \cdot \frac{(p_1 + p_2)}{(A_1 + A_2)^3}}}. \quad . \quad (145)$$

By measuring the areas and perimeters of two cross-sections, a distance l apart, and also the drop h in this distance, we may by formula (145) approximately compute the quantity flowing in the channel. The values of R_m and n (also $\frac{h}{l}$ if Kutter's C is being sought) are used in selecting a proper value for C.

If A_1 be made equal to A_2 in (145), the formula reduces to

$$Q = A\,C\sqrt{RS}$$

as one would expect.

It is common practice among engineers to solve problems in non-uniform flow by using the formula

$$Q = A_m C \sqrt{R_m S}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (146)$$

in which A_m and R_m are average values obtained by either of the methods mentioned above. The same formula may be derived from the general equation (144) by neglecting the difference in $\frac{{v_1}^2}{2g}$ and $\frac{{v_2}^2}{2g}$ and cancelling them from the equation. There results

$$h = \frac{l\,{v_m}^2}{C^2 R_m}$$

or

$$v_m = C\sqrt{R_m S}$$

and

$$Q = A_m C \sqrt{R_m S}.$$

Algebraic simplicity is doubtless obtained from such a method, but the result is rendered even more approximate than obtained from (145). It is true that the difference in the velocity heads at the end sections is small, but so is the quantity $\frac{h}{l}$, and results computed from (145) and (146) generally differ largely.

It the value of Q be known, also the shape and area of a section, then the depth at a subsequent section may be obtained by a method of trial, using equation (145). It will be necessary to first *assume a depth* at the second section from which all the quantities in the right-hand member of (145) may be determined. The value of Q as obtained from the equation is then compared with the given Q. If the former be too large, then the assumed depth at the second section must be increased until the computed Q agrees with Q as given. Likewise, if the computed Q is less than the given Q, the assumed depth must be decreased until a parity between the two is obtained.

131. Non-uniform Flow in Natural Channels.—In the foregoing discussion it was assumed that we were dealing with artificial channels possessing some degree of regularity in shape of cross-section, and having a constant bed slope within the length l. Also the roughness of the lining was supposed to remain constant. It will be seen that non-uniform flow in natural streams offers none of these characteristics. The sections are very irregular, the bed is most uneven in its slope, and the roughness of bed and banks may be continually changing. The application of the principles of the previous article to such streams must be attended by much uncertainty as to the results obtained, and determinations of Q should be preferably made by direct measurements as described in a later article. A certain problem, however, often arises in natural streams which cannot be solved by direct measurement, and because of its interest and importance will be treated in the following article.

132. Backwater.—If a channel be obstructed by a dam, weir or other construction, the raising of the latter will cause the water to set back up the stream, increasing the depth, and the flow in the portion of the stream so affected will become non-uniform.

It will differ from the particular problem previously cited, only in that the depth will increase in the direction of flow. Fig. 120 shows the conditions, *ab* being the original surface of the stream and *ef* the new surface or *curve of backwater*. Two important questions arise by reason of possibility of damage being done to property in time of flood, or even by the normal level of the backwater.

1. How much will the water be raised at a given distance upstream from the point of obstruction?

2. How far upstream will the influence of the backwater be felt?

Formulas purporting to answer these questions have been formulated and given in many text and reference books on

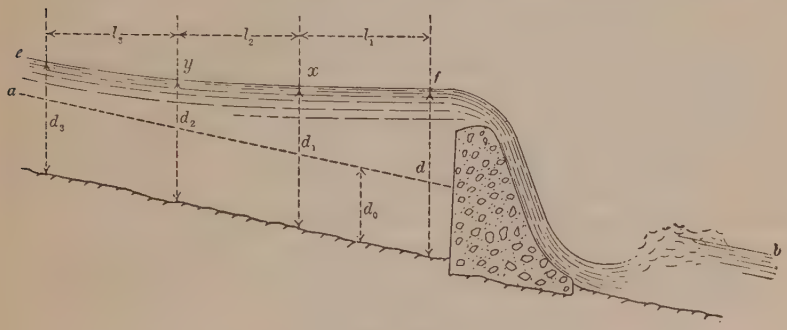


FIG. 120

hydraulics, but they have been based largely on the assumption of a *regular* channel having a *uniform slope to its bed*. Obviously a natural stream does not have these characteristics and the formulas apply with less certainty in result, the farther the departure from such conditions. The method given, therefore, will be one which, although necessarily approximate, will contain all the accuracy that will usually be found in the given data.

Of the several difficulties encountered, two are the irregularity of the cross-sections and the variation in the roughness of the channel from section to section. These make it impossible to estimate values for C and R that will hold good throughout the length of stream affected by the backwater. We will, therefore, imagine the stream to be divided into sections of lengths l_1, l_2, l_3,

etc., so chosen that the values of A, R and n throughout any section vary as little as possible for that section.

Commencing with that section which lies nearest the dam where the elevation of the water and the actual cross-sectional area are known for the condition of maximum flow, we will, by the method of trial, ascertain the probable level of the surface at the upstream end of the section by computing the surface slope in this distance necessary to discharge the given quantity. This having been obtained, the next section is similarly treated and by a series of such computations we may arrive at the point where the curve of backwater practically coincides with the original surface of the stream. The profile of the curve is thus obtained and both of our questions answered.

To obtain the data necessary for the computations, a survey of the stream should be made and the shape of each cross-section, up to the probable level of maximum flow, determined. Preferably these should be plotted so that areas at each section may be planimetered and the perimeter measured for any water level. The survey should also include the collection of such data regarding the bed and banks as will make the selection of a proper value of n for each section possible. Close values for n may also be obtained by a direct meter measurement of the *present* discharge (see Art. 136), allowing n to be computed for each length from

$$Q = A_m C \sqrt{R_m S}.$$

A_m and R_m are the average area and hydraulic radius for each reach or section, and S is the surface slope of the section obtained by running a line of levels up the stream. The equation furnishes a value of C from which the corresponding n is determined.

The detailed computation for the slope of each reach being the same, only those for the first reach will be outlined.

Since the discharge, Q, is known, and the depth of the water immediately behind the dam, the slope of the stream from x to f will be tentatively obtained by assuming the elevation of the surface at x. From the plot of the cross-section at x, the value of A_x may be determined, also R_x. The data are then similar to that given in Art. 130, and equation (145) may be used to ascertain what quantity would be discharged by the reach under the assumed slope. If this be in excess of the given amount, Q, then the eleva-

tion of x has been assumed too great and a smaller value must be assumed and the corresponding discharge computed, the process being repeated until the discharge as computed from the assumed elevation of x agrees with the given discharge.

The elevation at x having thus been determined, the upstream elevation at the end of the next reach is computed in a similar manner.

Some engineers prefer using

$$Q = A_m C \sqrt{R_m S}$$

in place of equation (145), but the equation is really no simpler than (145) and is probably less accurate.

133. Other Losses in Head.—In addition to the loss due to channel friction, the losses which are comparable to those occurring in pipes are also found. Thus, at entrance to the channel, a slight loss occurs due to the fact that a drop in the surface, or gradient, is necessary to cause the water to enter the channel with the velocity which is to be maintained by the slope of the channel (Fig. 118). The loss may be assumed equal to $\frac{v^2}{2g}$.

Abrupt changes in cross-section bring about losses, but expressions for their magnitude cannot be as definitely framed as in the case of pipes. It is often assumed that the equations derived for these losses in pipes may be used for open channels.

Both natural and artificial channels may have many bends giving rise to losses in head, but no attempt has been made to formulate the latter.

Common practice is to recognize the existence of all the above losses and to give to the channel an extra compensating slope.

134. Distribution in Velocity at a Cross-section.—As in the case of pipe flow, the velocity past any section varies from point to point in the section, the forward component having its largest value in that portion most remote from the sides of the channel. For artificial channels having a symmetrical section, this point will be about equidistant from the sides and somewhat below the surface. As the sides and bottom are approached from this point, a gradual lessening of velocity is noted, the minimum being reached at the sides or bottom.

In natural streams the distribution of velocity is similar to that in artificial channels especially when the cross-sections are fairly regular.

In cross-sections that are irregular with respect to depth, the thread of maximum velocity is generally found in the vertical which marks the greatest depth. In any channel, the presence of obstructions or bends will largely affect the distribution of velocities.

Figure 121 shows the general relation which exists between the velocities as measured past a vertical at any point in the cross-section of a stream, and the curve *a-b* is called a *vertical velocity curve.* Its form in the figure indicates a smoothness and regularity that is not always found when a curve is drawn to connect the points plotted from observed velocities, but it is typical. From thousands of observations it has been shown that the curve is well represented by a parabola having its axis horizontal and somewhat below the surface.

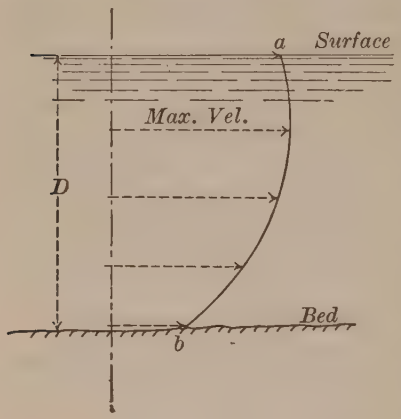


Fig. 121

Just why the thread of maximum velocity should be *below* and not *at* the surface has never been definitely determined; Stearns, as a result of experimental observations, offers the explanation that it is probably due to the fact that there is an upward movement in the water close to the sides of a channel which carries the slower water at this point upward to the surface where it flows obliquely outward toward the center of the stream, retarding the velocity of the surface layers.

The location of the thread of maximum velocity in any vertical has been found to vary considerably, sometimes being close to or at the surface, but more generally found below the surface at a distance equal to two or three tenths the depth.

Assuming the vertical velocity curve to be a parabola with horizontal axis, the position of the *mean* velocity in the vertical will be found to vary from 0.58 D when the maximum is at the

surface, to 0.65 D when the maximum is 0.3 D below the surface. This may be shown true by an analysis based on the mathematical properties of the parabola and, inasmuch as the maximum is ordinarily found between the surface and the 0.3 depth point, it may be assumed that the velocity found at 0.6 depth in any vertical, fairly represents the mean velocity in that vertical.

Again, assuming that the velocity curve be a parabola with axis horizontal, it may be demonstrated that, irrespective of the location of the axis (maximum velocity), the mean velocity in a vertical is closely represented by the arithmetical mean of the velocities found at $\frac{2}{10}$ and $\frac{8}{10}$ depth. That is

$$\text{Mean Velocity} = \frac{\text{vel. at } \frac{2}{10} \text{ depth} + \text{vel. at } \frac{8}{10} \text{ depth}}{2}$$

These facts, or relations, are utilized in determining well established methods for measuring the discharge of streams by current meters, as explained in a later article.

135. Energy; Alternate and Critical Stages; Hydraulic Jump.—The amount of energy in the water passing any cross-section of a channel may easily be computed with reference to a given datum. Thus in Fig. 122, the total *energy per pound* in the water at any point n may be written (see Art. 39),

$$E = \frac{v^2}{2\,g} + \frac{p}{w} + z,$$

or

$$E = \frac{v^2}{2\,g} + D, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (147)$$

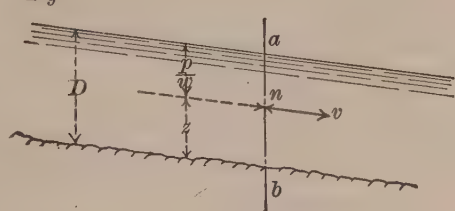


FIG. 122

if the datum for computing z be taken through the base of the section. If to v be given a numerical value which is the average velocity of all particles crossing the section, then (147) will give the average value of E for all water passing a-b.

Since $W = wAv$, the amount of energy passing the section *per second* is found from

$$\text{Total energy per second} = W\left(\frac{v^2}{2g} + D\right). \quad . \quad . \quad . \quad (148)$$

If the datum be assumed through the surface of the water, the average value of $\frac{p}{w}$ will be $\frac{D}{2}$, and the average z will be $-\frac{D}{2}$, giving

$$\text{Total energy per second} = W\frac{v^2}{2g}, \quad . \quad . \quad . \quad . \quad (149)$$

which is the kinetic energy alone of the stream. It should be noted that, if v represent the arithmetical mean of all the different velocities past a section, the value of v^2 is not the same as the mean of the squares of the separate velocities from which v is computed. Stated in mathematical terms, the square of the mean is not the same as the mean of the individual squares. To be strictly correct we should write

$$\text{Mean velocity head} = k\frac{v^2}{2g},$$

k having a value slightly greater than unity. No serious error ordinarily arises if k be assumed equal to unity, and it will be noted that this has been done in all previous discussions in this book.

With *uniform* flow, equation (147) shows that the energy content of the water is constant from section to section if the datum be taken through the base of the section, and the loss in energy between any two sections is represented by the drop in the channel, if a common datum be taken for both sections.

In a paper presented before the American Society of Civil Engineers, November 3, 1915, Mr. Karl R. Kennison, demonstrated that in an open channel where a fixed rate of flow is being maintained, there is, in addition to the existing water level, another level at which the same quantity of water might be made to flow and have the same energy content. The proof is simple and will be borrowed from Mr. Kennison's article.

From equation (147),

$$E = \frac{v^2}{2g} + D.$$

Assuming D constant and a rate of flow, Q, per foot of width of stream, we have

$$v = \frac{Q}{D};$$

$$E = \frac{Q^2}{D^2 2g} + D, \quad . \quad . \quad . \quad . \quad . \quad . \quad (150)$$

or

$$D^3 + \frac{Q^2}{2g} - ED^2 = 0. \quad . \quad . \quad . \quad . \quad . \quad (151)$$

For given values of Q and E, this cubic equation is satisfied by three values of D, two of them being real, and one imaginary. That is, there are two depths at which a given rate of flow may be maintained with the same energy content. To illustrate, let it be assumed that in a certain channel there is flowing, per foot of width, a quantity of 50 cu. ft. per second at a depth of 9.58 feet. By equation (150),

$$E = \frac{50^2}{9.58^2 \times 64.4} + 9.58 = 10 \text{ ft. lb. per lb.}$$

which substituted in (151) gives

$$D = 9.58 \text{ or } 2.24 \text{ or } -1.82 \text{ feet.}$$

The water may flow at a 9.58 ft. depth with a velocity of 5.22 ft. per second, or at a 2.24 ft. depth with a velocity of 22.4 ft. per second. The first condition Kennison describes as the *Upper Alternative Stage,* and the second as the *Lower Alternative Stage.* The upper stage with its low velocity has a depth practically the same in value as the energy per lb. in the water. The lower stage with its high velocity is unstable in its condition as has been proved by observation and experiment. The two stages are often seen occurring just below the spillway section of a dam (Fig. 123). If the spillway section has the proper shape so that the water be discharged at the toe of the spillway in a shallow stream at high velocity, it will be noticed that the level of the stream's surface is much lower than the level of the channel below the dam, and that after a short distance has been passed, the water suddenly changes into a tumbling mass and rises to the level of the down-stream channel by a standing wave or jump. The jump is simply the transition from the lower high-velocity stage to the more stable upper stage.

Either stage, once inaugurated, may be maintained if the slope of the channel be correct to maintain the accompanying velocity. The problem becomes more interesting if the relation between energy and depth be developed for the condition that the quantity

FIG. 123

flowing shall be a maximum for the given energy content. To do this, equation (151) may be solved for Q giving

$$Q = \sqrt{2\,g\,(ED^2 - D^3)}.$$

If $\frac{\delta Q}{\delta D}$ be computed and placed equal to zero, the value of D, in terms of E, may be found that will insure Q being a *maximum* for a given energy content. There will result,

$$D = \frac{2}{3}E, \quad \ldots\ldots\ldots \quad (152)$$

and the corresponding Q will be

$$Q = 3.09\,E^{\frac{3}{2}}. \quad \ldots\ldots\ldots \quad (153)$$

In explanation of these results let us assume the value of Q at 50 cu. ft. per sec. as in the previous illustration. From (153) the value of E will be found to be 6.40 ft. lb. per lb. (or ft.) and from (152) the value of D is 4.27 ft. We have found a new depth at which 50 cu. ft. per second may flow, but the energy content which was 10 ft. lb. per lb. in the previous illustration is now only 6.40 ft. lb. per lb. (based on a datum through the bottom of the cross-

section). Fig. 124 shows how this condition might be brought about. The stream is flowing at the upper alternative stage with a depth of 9.58 ft. and an energy content of 10 ft. lb. per lb. If the bed of the channel be given a gradual elevation of 3.60 ft., the quantity of 50 cu. ft. per second may be made to flow over the new bed at a depth of 4.27 ft. and with an energy content (relative to the new bed) of 6.40 ft. lb. per lb. Its energy content relative to the old bed will still be 6.40 + 3.60 or 10 ft. lb. per lb.

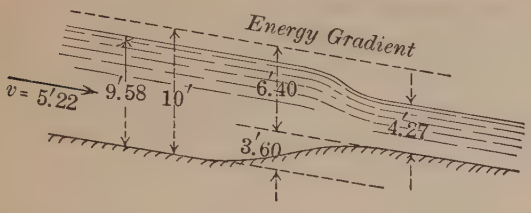


FIG. 124

If the value of 6.40 be substituted for E in the general equation (151), and 50 for Q, the solution for the value of D will give two positive roots which are numerically identical and equal to 4.27 ft., showing that for these values of E and Q there is only one possible depth. It appears therefore that a stream carrying a given quantity of water may have two stages or one, depending on whether or not the discharge is a maximum for the energy content. If it be a maximum, then only one stage, which we will call the Critical Stage, is possible. The name is given for the reason that if the given quantity be made to flow at a less depth, the flow will be at the lower alternative stage and, if not maintained by a smooth channel having the proper slope, will finally jump to the upper stage. The critical stage also marks the merging of the two alternative stages into one from which, if conditions are right, the flow may emerge in either one of the two stages. The immediate cause of the jump may be a change in the channel or a lessening of the energy content by reason of the excessive friction loss accompanying the high velocity of the lower stage.

By combining

$$\frac{v^2}{2g} + D = E$$

and

$$D = \frac{2}{3}E,$$

we obtain

$$v = \sqrt{g\,D}$$

as the velocity of flow at the critical stage. The immediate foregoing statements may now be repeated by saying that when water flows in a channel with a velocity greater than $\sqrt{gD}$, conditions are favorable for the formation of a jump.

The accompanying graph of equation (151), drawn for $Q = 50$ cu. ft. per sec., will be helpful in visualizing the principles which have been discussed. The curve is plotted by assuming values of D and calculating corresponding values of E. For $E = 10$, two

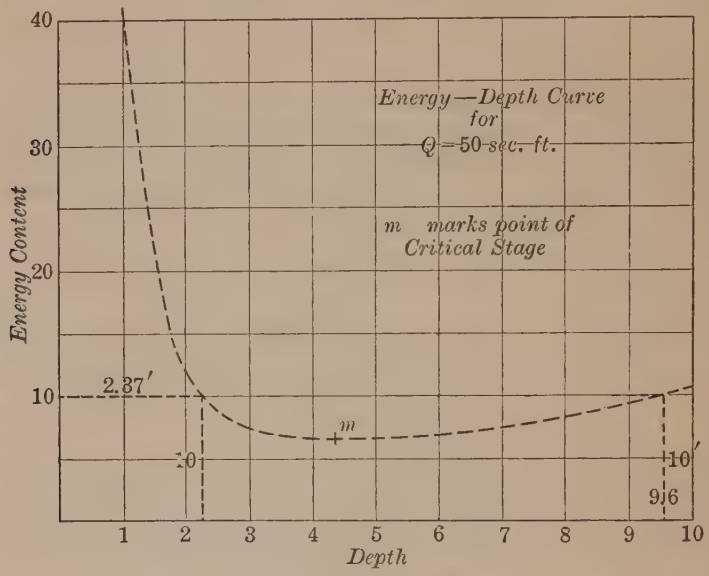


FIG. 125

values for D are noted on the curve, equal to 2.37 and 9.60 ft., these being the alternate depths for this energy content, as previously determined. The lowest point on the curve m, has co-ordinates of $E = 6.40$ and $D = 4.27$ which we have found as the conditions for a discharge of 50 ft. per sec., under a minimum E at the critical stage. Also it can be seen that with flow taking place at this stage, any slight change in the value of E will result in a decided change in the water depth.

The occurrence of the hydraulic jump is always accompanied by a loss of head caused by the sudden reduction in velocity while passing through the jump. Where the difference in depth before and after passing the jump is small, the loss is probably

insignificant and the jump takes place smoothly with a noticeable absence of eddying. The greater the height of the jump, the greater becomes the turbulence and the resulting loss in head.

The loss in head causes the energy content of the stream to be reduced, and prevents the water level from reaching the same upper stage that theoretically existed before the water entered the jump.

Woodward, in his Theory of the Hydraulic Jump (Miami Conservancy District Reports, Part III), offers the following derivation of the relation between the depth before and after passing the jump.

Let D_1 = depth before entering jump.
D_2 = depth after passing jump.
V_1 = velocity of stream with depth D_1.
V_2 = velocity of stream with depth D_2.
D_c = critical depth.
V_c = velocity of stream with depth $D_c = \sqrt{gD_c}$

$$\text{Mass of water flowing per second} = \frac{wQ}{g}$$
$$\text{Change in velocity} = V_1 - V_2.$$

By the principles of Art. 138, the resultant of the external forces acting on the stream (and parallel to its motion) to change its velocity, is equal to the product of the mass per second flowing in the stream and the change in velocity which occurs. This product is

$$\frac{wQ}{g}(V_1 - V_2).$$

The forces acting on that portion of the stream which includes the jump and the change in velocity, are,—

(1) The static pressure of the water on an imaginary vertical cross-section of the stream where the depth is D_1.

(2) Corresponding pressure on an imaginary cross-section beyond the jump where the depth is D_2.

(3) The frictional resistance to motion offered by the bed of the stream. This is small and may be neglected.

The first force, acting in the direction of motion has a value, $\frac{w D_1^2}{2}$.

The second force, opposing the first, has a value, $\frac{w D_2^2}{2}$.

Their resultant is $\frac{w}{2}(D_2^2 - D_1^2)$, and by the principle previously stated,

$$\frac{w}{2}(D_2^2 - D_1^2) = \frac{wQ}{g}(V_1 - V_2).$$

Since $V_2 = V_1 \frac{D_1}{D_2}$, this may be written

$$\frac{(D_2^2 - D_1^2)}{2} = \frac{Q}{g}\left(V_1 - V_1 \frac{D_1}{D_2}\right) = \frac{QV_1}{g}\frac{(D_2 - D_1)}{D_2},$$

or

$$\frac{D_2 + D_1}{2} = \frac{QV_1}{gD_2}.$$

If we substitute $\frac{Q}{D_1}$ for V_1, we obtain

$$D_1 D_2 \frac{(D_1 + D_2)}{2} = \frac{Q^2}{g} = D_c^3 \text{ (since } Q = D_c\sqrt{gD_c}).$$

This equation relates D_1 and D_2 for any rate of flow, Q. The equation shows that if D_1 equals D_c, then D_2 also equals D_c (this we have already shown). If a jump is to occur, D_1 must be less than D_c, and the jump, when it occurs, takes place across the critical depth.

The discussion contained in this article is necessarily brief as the book is intended for an elementary text, and the student is referred to Mr. Kennison's excellent paper, and its many contained references, for a more elaborate treatment of the subject.

136. Measurement of Flow in Natural Channels.—From what has already been said in discussing flow in artificial channels, where regularity of section, lining and slope is generally present, it is at once apparent that it becomes a most difficult and probably impossible task to formulate the law of flow in channels where these same factors are constantly changing. While much work has been done in this very important field of hydraulics, all efforts to originate an accurate, or reasonably accurate, formula which will apply to streams of all kinds and sizes have so far come to naught. It is often claimed that the Chezy formula using the Kutter coefficient may be applied to natural streams, but it is beyond reason to expect that anything more than mere approximations may be obtained from any formula. When it is desired to ascertain the rate of flow, dependence must be placed on direct measurements, made in the stream, from

which the discharge may be computed. Several methods are in general use. For small streams it is possible, and often practicable, to construct a weir and obtain very good results. In larger streams it is too expensive a method and generally impracticable. Here the most common practice is to employ a current meter which enables the observer to determine with remarkable accuracy the velocity at any point in the stream at which the meter may be placed. Fig. 126 illustrates the Price current meter as developed and used in this country by the engineers of the Geological Survey. It consists of a wheel fitted with cupped vanes and mounted on a vertical axis about which it is free to turn under the action of the moving water. The whole is supported on the upstream end of a horizontal shaft which at its other end is fitted with directional vanes which steady the meter and keep it headed into the current. As illustrated, the meter is intended for use in shallow water where wading is possible, and it is supported by a vertical rod held in the hands of the observer. In deeper water, the rod is removed and the horizontal shaft is pivoted at its center to a short vertical stempiece. The latter, at its upper end, has a small hole for connecting a cord by which the meter is suspended from a boat or bridge. The lower end of the stempiece fastens to a lead weight which steadies the meter and holds it in position. The wheel revolves at a rate proportional to the velocity of the current in which it is placed, and by "rating" the meter it is possible to determine the relation between the revolutions per second and the velocity of the water in feet per second. Rating is accomplished by moving the meter through still water at different velocities and

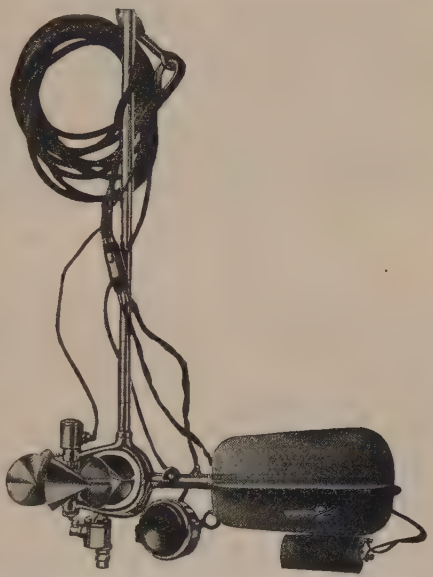


Fig. 126.—Price Current Meter as Manufactured by W. & L. E. Gurley, Troy, New York.

noting the corresponding rates of revolution. A rating curve is then constructed and, if desired, a rating table made. With the type of meter shown, it is found that the rating curve is practically a straight line save at very low or high velocities. It has been shown that slightly different results are obtained if the rating is done by holding the meter stationary in a stream of water moving at a known velocity, but the difference is not great and the usual practice is to move the meter through still water. With the meter immersed, the revolutions of the wheel are counted by means of an electric circuit which is broken at each revolution by means of a commutator attached to the wheel shaft. The wires of the circuit pass from the meter to the surface where a small buzzer, ear-phone or counter records the make and break of the circuit. Current is furnished by a small dry cell battery. Usually the wires of the circuit are utilized as a supporting cord for the meter. Suspension cord, ear-phone and battery are shown in Fig. 126.

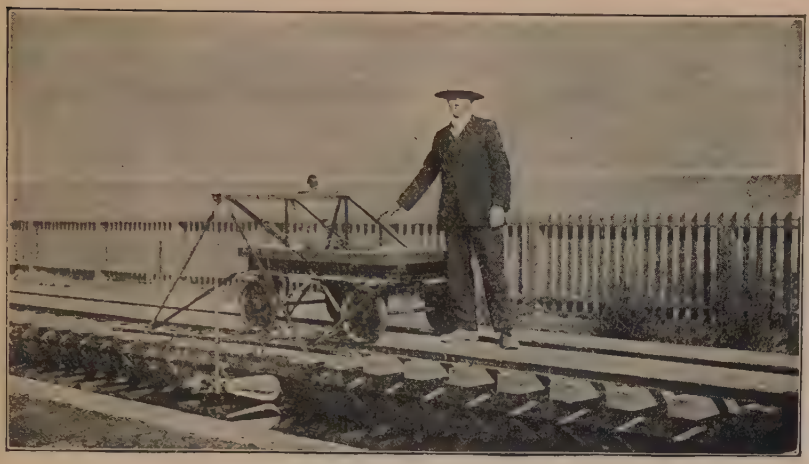
The discharge of a stream being the product of its cross-sectional area and the mean velocity of the water past the section, a meter measurement consists of determining with all possible accuracy the value of these two factors. The area for any stage of flow may be easily determined by soundings made across the selected section, the distance of each point of sounding being measured from a permanent point on the bank and in the line of the cross-section. If desired, the profile of the section may be plotted on paper and the area measured for any flow-level or stage. It has been found that best results are obtained from gaugings if the area be subdivided into a series of vertical strips, preferably of equal width, and the discharge past the entire section be computed on the basis that

$$\text{Total } Q = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + \text{etc.},$$

a being the area of a strip, and v the mean velocity for the strip.

To obtain the value of v, it is commonly assumed that the mean velocity as measured in a vertical at the middle of a strip may be taken as the mean velocity for the entire strip. Various methods for obtaining the mean velocity in a vertical are as follows:

Multiple Point Method.—The velocity is observed at a number of points spaced about equally between the surface and the bottom, points close to the surface and bottom being usually included. If a sufficient number of points be chosen, the arith-



Canadian Meter-Rating Station

Gaging-Station Utilizing a Footbridge over a Small Stream

metical mean of the observed velocities may be taken as the true mean, or a curve of vertical velocities (see Art. 134) may be plotted and the mean computed as equal to the area within the curve divided by the depth.

Two and Eight-Tenths Method.—This consists of determining the velocities at $\frac{2}{10}$ and $\frac{8}{10}$ the depth and assuming the mean velocity in the vertical equal to their arithmetical mean as explained in Art. 134.

Six-Tenths Method.—A saving of time is effected if but one observation be made in each vertical at $\frac{6}{10}$ the depth, and the velocity there found assumed to be the true mean. The basis of this assumption is explained in Art. 134.

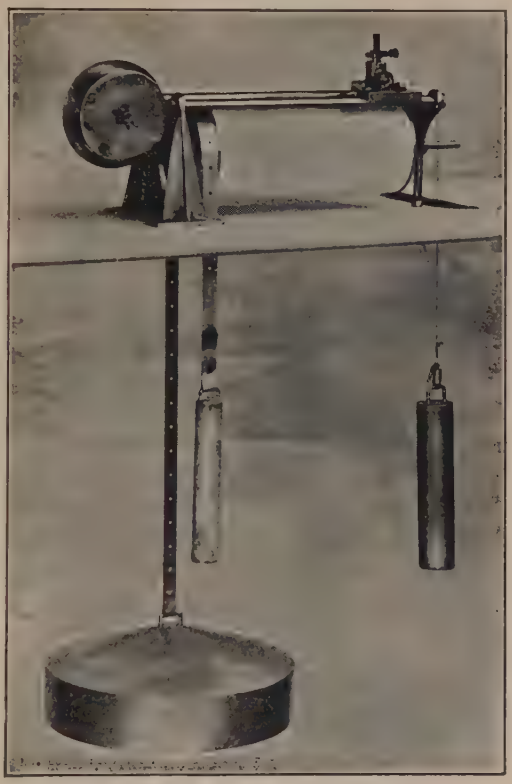
Integration Method.—If the meter be moved slowly from the surface to the bottom and the total number of revolutions noted, together with the time elapsing in making the descent, the average revolutions per second may be computed and a corresponding velocity obtained which will be assumed to be the true mean. The reasoning behind the method is apparent. If the rate of descent be uniform, the meter will be exposed to the various velocities for equal lengths of time and the average rate of revolving should closely correspond to the average velocity. It is not essential that the movement of the meter be made from the surface downward. It may be reversed or even moved continuously from top to bottom and back to the top.

Comparison of Methods.—The first three methods named are given in the order of their generally accepted accuracy. The integration method is probably equal in accuracy to either of the first two, but it requires a skillful operator. The multiple point method is based on no assumption save that a sufficient number of observations are made to warrant the arithmetical mean being taken as the true mean. The two and eight-tenths method assumes the vertical velocity curve to be a parabola, while the six-tenths method assumes that the parabola has its vertex somewhere between the surface and $\frac{3}{10}$ the depth.

The two and eight-tenths method is much used by the engineers of the Geological Survey as it is rapid and, in a majority of cases, gives excellent results. It has been shown suitable for use in ice-covered channels.

Selection of Section.—In selecting a site for gauging the flow of a stream, it is desirable to choose a fairly symmetrical section

at the end of a long straight reach where disturbances and eddyings will be a minimum. The section should be free from gross irregularities and, if measurements are to be made over an extended period of time, the bed and banks should not be subject to scouring. Where it is desired to keep a daily or weekly record of the stream flow it is customary to make record of the elevation of the surface at the time of making the soundings, and determine the depths at any later time by adding or subtracting the change in level that may have occurred in the meantime. This necessitates the constructing of a gauge which will show at any time the level at the section. The gauge may consist of a painted board marked to hundredths of a foot and placed where it will not be dislodged, or it may be an automatic recording device consisting of a float which moves a recording pencil over paper actuated by clockwork. Other gauges have been devised and will be found described in River Discharge by Hoyt and Grover, John Wiley & Sons, N. Y., as well as a complete description of the theory and practice of stream measurements. Where the rise and fall of the surface level be assumed as indicative of the amount flowing in the stream, it is absolutely necessary that this level be unaffected by backwater from below.



Automatic Water Level Recorder Manufactured by W. & L. E. Gurley, Troy, N. Y.

This may be accomplished by selecting a site above a fall or rapids.

Computation of Discharge.—Once the cross-section has been divided into strips and the mean velocity at each strip determined by one of the methods above outlined, the computation of discharge at once is possible by a summation of the partial discharge through each strip.

137. Other Methods of Gauging.—The Salt Velocity Method which has been explained in Art. 117 may be used in small channels provided suitable electrodes can be devised and installed. At the present writing its use has been confined principally to closed conduits, but the writer believes that it may be adapted with good results to open channels. Further experimental work is necessary before the limits of its application are known.

A method somewhat similar but differing in principle, known as the Salt Dilution Method, has been in use for some time and has been proven very satisfactory from the standpoint of accuracy. It consists of adding a concentrated solution of salt to the flowing stream and by analysis determining its dilution after traversing a length of stream sufficient to cause a uniform mixture with the water of the stream. The solution is added at a constant rate and no measurements of area or distance are necessary.

Let W = pounds of water discharged per second by stream.
W' = pounds of salt solution added per second to the stream.
R = percentage of natural salt (by weight) flowing in the stream to be gauged.
R' = percentage of salt in the concentrated solution.
R'' = percentage of salt in samples taken from stream after thorough mixing has taken place.

The pounds of salt, passing per second the lower point on the stream where samples are taken, must equal the sum of the pounds of the natural salt carried by the stream and the pounds of salt added in the concentrated solution or,

$$W R + W'R' = (W + W') R''.$$

The values of the three R's being known by analysis the equation may be solved for W.

The inclusion of R in the equation is necessary inasmuch as many streams have a natural salt content which might be sufficiently high in percentage to introduce a considerable error if it were neglected.

Experimental work done at the Massachusetts Institute of Technology and elsewhere, has shown that the method may be expected to give a high degree of accuracy when carefully applied. On large streams it has been found that excessive quantities of salt are required and the apparatus for dosing and taking samples becomes expensive to install. The method has been successfully used in the testing of hydraulic turbines after installation, the turbine providing excellent means for thoroughly mixing the solution with the stream.

PROBLEMS

1. A canal, in firm clean earth, has a bottom width of 20 ft. and side slopes of 2 horizontal to 1 vertical. If the water depth be 4 ft. and the slope of the surface 1 in 1500, compute the probable velocity and discharge. Use Kutter's C. *Ans.* 3.9 ft. per second.
437 cu. ft. per second.

2. A trapezoidal canal is to have a base width of 20 ft. and side slopes of 1 to 1. The nature of the soil (clean earth) limits the velocity of flow to 2 ft. per second. What slope must be given the bed in order to deliver 182 cu. ft. per second? Use Kutter's C. *Ans.* $s = 0.00027$.

3. How deep will water flow in a 14-foot rectangular channel that is carrying 615 cu. ft. per second, if $s = 0.00075$ and the channel be lined with smooth cement? Use Manning's C. *Ans.* 5.2 ft.

4. A circular brick conduit 3 ft. in diameter flows half full with a slope of 1 in 2000. Compute its probable discharge using (*a*) Bazin's coefficient; (*b*) Kutter's coefficient; (*c*) Manning's coefficient.

5. What slope would it be necessary to give a rectangular channel 50 ft. wide and 10 ft. deep in order to discharge 1500 cu. ft. per second? Assume ordinary timber lining.

6. A trapezoidal canal is to carry 1600 cu. ft. per second with a mean velocity of 2 ft. per second. One side is vertical, the other has a slope of 2 horizontal to 1 vertical, and the lining is rubble masonry. Compute the minimum hydraulic slope. Use Manning's coefficient.

7. A trapezoidal canal of symmetrical form with side slopes of 1½ horizontal to 1 vertical is to carry 500 sec. ft. with a velocity of 2.5 ft. per second. What is the minimum amount of lining (in sq. ft.) required per foot length of canal?

8. An open channel of symmetrical form is to contain 150 sq. ft. of wetted cross-section and have sides sloping 2 horizontal to 1 vertical. If given most favorable proportions, what will the channel discharge if the hydraulic slope be 1 in 2500? Assume $m = 1.54$ and use Bazin's coefficient.

9. A rectangular channel is to carry 75 cu. ft. per second on a slope of 1 in 10,000. If lined with smooth stone ($n = 0.13$), what dimensions ought it to have if the wetted perimeter is to be a minimum? Use Manning's coefficient.

10. A rectangular channel, 18 ft. wide and 4 ft. deep, has a slope of 1 in 1000, and is lined with good rubble masonry. ($n = 0.017$.) It is desired to increase as much as possible the amount discharged without changing the channel slope or form of section. The dimensions of the section may be changed but the channel must contain the same amount of lining as the old. Compute the new dimensions and probable increase in discharge. Use Kutter's coefficient. *Ans.* 13 ft. by 6.5 ft.; 118 cu. ft. per second.

11. A canal is to have a trapezoidal section with one side vertical and the other sloping at 45 degrees. It is to carry 900 cu. ft. per second with a mean velocity of 3 ft. per second. Compute the dimensions of the section which would require a minimum hydraulic slope.

Ans. Base width 17.7 ft; depth 12.5 ft.

12. A triangular-shaped channel is to be designed to carry 24 cu. ft. per second on a slope of 0.0001. Determine what vertex angle and depth of water over the vertex will be necessary to give a section with minimum perimeter, assuming the channel to be built from timber planking. Use Manning's coefficient.

13. A trapezoidal channel with side slopes of 45° is to have a water depth of 10 ft. and carry 1600 cu. ft. per second with a mean velocity of 2 ft. per second. What dimensions should it have for best proportions?

14. A rectangular timber flume, 10 ft. wide by 5 ft. deep, discharges 200 cu. ft. per second. If the same material had been used for a trapezoidal section having the same perimeter and side slopes of 1 to 1, what would have been the greatest discharge possible with the same hydraulic slope?

15. A rectangular channel, 18 ft. wide by 4 ft. deep, is lined with smooth stone, well laid, and has a hydraulic slope of 0.001. What saving in earth excavation and lining, per foot of length, could have been effected by using more favorable proportions, but adhering to the same discharge and slope?

16. In designing a canal for supplying a power plant the problem arises as to whether a trapezoidal or rectangular section should be built. If the former be used, it is found that, with a clean earth lining, a section having a top width of 40 ft., bottom width 10 ft., and water depth 5 ft., will deliver the required quantity of water if laid on a slope of 0.75 per 1000. Could a less slope be used by employing a rectangular section, lined with rubble masonry, if the area and velocity of flow be maintained at the same figures as in the trapezoidal section? What would its value be? Use Kutter's coefficient. *Ans.* $s = 0.00032$.

17. Two circular conduits, each 5 ft. in diameter, serve to carry the waters of a creek through a railroad embankment. When carrying flood discharges both ends of the conduits are submerged. Assuming no change in the hydraulic gradient or in the value of C, what width would be necessary in two equal rectangular sections, each 4 ft. deep, if they are to replace the circular conduits and perform the same service? *Ans.* 5.2 ft.

18. Two ponds with a difference in level of 0.05 ft. are connected by 4000 ft. of rectangular canal, lined with smooth cement. The canal is 20 ft. wide, laid on uniform grade, with water 8 ft. deep at its upper end and 10 ft. at the lower end. Compute the rate of discharge.

19. A rectangular canal 6000 ft. long, 10 ft. wide, is carrying 80 cu. ft. per second. The canal floor is level and at its lower end is fitted with a suppressed weir 3.5 ft. high. Assuming $n = 0.017$ and that the stream surface is a plane, compute the depth of water necessary in the canal at its upper end in order to maintain the given discharge. *Ans.* 5.8 ft.

20. In a length of 3000 ft. the surface of a trapezoidal canal has a fall of 1 ft., the water being 5 ft. deep at the upper section and 4 ft. deep at the lower. The base width is 10 ft. and the side slopes are 2 horizontal to 1 vertical. If the nature of the soil indicates a value for n of 0.025, compute the probable rate of discharge.

21. A rectangular canal 50 ft. wide carries 2000 cu. ft. per second. The depth is 10 ft. at a certain section, and the floor of the canal has a uniform slope of 1 in 10,000. What depth would be probable at a distance 4000 ft. upstream from the mentioned section? The bed of the canal is

earth and the side walls are rubble masonry giving an estimated value of 0.022 for n.

22. A channel whose section is approximately rectangular discharges 30 cu. ft. per second *per foot of width*. The depth being 7.5 ft., compute the possible alternate stage and the corresponding velocity.

23. A canal is to discharge 20 cu. ft. per second, per foot of width, with a minimum energy content. Compute the value of the latter and the corresponding depth. If the channel be 40 ft. wide (rectangular) and $n = 0.017$, what slope would be necessary to maintain this depth and rate of discharge?

24. A channel carrying water with a velocity of 15 ft. per second has a uniform depth of 3.7 ft. (*a*) Show by computations at what stage the flow is taking place. (*b*) Is a jump possible, and if so, what would be its approximate height?

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CHAPTER X

DYNAMIC ACTION OF JETS AND STREAMS

138. Principle of Momentum.—In this chapter will be discussed the action of moving streams with regard to the forces which they exert upon bodies that constrain them in their motions. A few fundamental principles in mechanics are involved and will be briefly reviewed.

If a mass be acted upon by a constant force, it suffers an acceleration in the direction of the force which may be determined from the relation $P = Ma$. The acceleration being uniform,

$$a = \frac{\text{Change in Velocity}}{\text{Time}} = \frac{\Delta v}{\Delta t},$$

and

$$P = M \frac{\Delta v}{\Delta t}.$$

If the change in velocity be that which occurs in one second, then

$$P = M \Delta v, \quad . \quad . \quad . \quad . \quad . \quad . \quad (154)$$

and the change in Mv, or *change in momentum*, is equal to the value of the constant force which produced it, or the *average* value if the force is not constant.

It is to be particularly noticed that v is a vector quantity having direction as well as magnitude, and Δv represents the *vector* change in velocity.

139. Application to Fluid Masses under Conditions of Steady Flow.—As an aid to the use and understanding of the above principle when dealing with fluid masses, the following demonstration will be found helpful. Figure 127 shows a portion of a gradually enlarging channel containing water in steady motion. As a consequence of the gradual change in sectional area, the velocity of the contained water particles, and hence their momentum, is gradually reduced. The velocity at some section m we may assume to be v_1, while at n it has been reduced to v_2. Suppose that in the time dt particles at m move to m' and those at n to n',

so that the distance mm' equals v_1dt and nn' equals v_2dt. The aggregate momentum of all the particles between m and n suffers a diminution which may be represented in the difference found between the momentum of the mass mm' and nn'. That this is so may be seen if it is noted that the aggregate momentum of all particles between m' and n remain constant. (The student should

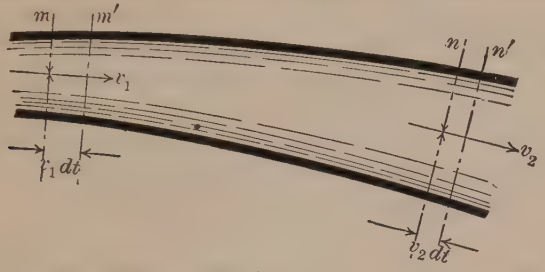

FIG. 127

here review the discussion on page 60 relative to the change in kinetic energy taking place in the mass A-B, Fig. 39.) The momentum of the mass mm' being $\left(\frac{wa_1v_1dt}{g}\right)v_1$,

and that of nn', $\left(\frac{wa_2v_2dt}{g}\right)v_2$, we have from Art. 138,

$$P = \left(\frac{wa_1v_1^2dt}{g} - \frac{wa_2v_2^2dt}{g}\right) \div dt,$$

or, since

$$a_1v_1 = a_2v_2 = Q, \text{ and } \frac{Qw}{g} = M,$$

$$P = M\,(v_1 - v_2).$$

Here $(v_1 - v_2)$ represents the vector change in velocity and should be replaced by Δv giving

$$P = M\,\Delta v. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (155)$$

The equation is identical with (154) as far as notation is concerned but it is different in several respects. Equation (154) deals with a *rigid* body of mass M, whose velocity is changed in *one second* by an amount Δv. Equation (155) deals with a *liquid stream* whose velocity between any two sections is changed by Δv in no particular time. The time element enters equation (155) in computing M, which is the mass flowing per second past a cross-

section of the stream. The equation may for convenience and emphasis be stated as follows:—

In any stream of steadily moving water, the unbalanced force acting on the stream between any two sections, changing its velocity by Δv, may be computed by multiplying the mass of water flowing per second in the stream by the vector change in velocity occurring between the two sections.

If a component of the unbalanced force in a direction parallel to a designated axis be desired, it may be computed by calculating Δv as the change in the corresponding component of velocity between the two sections. These principles are *fundamental* in all problems dealing with the dynamic effect of flowing liquids and will be freely used in the following discussions.

140. Force Exerted by a jet upon a Deflecting Surface.—If a stream or jet be turned from its path by meeting tangentially a deflecting surface (Fig. 128), it exerts upon the surface a dynamic pressure. We may consider the equal and opposite force, P, to have been the cause of the deflection and measure it as follows:

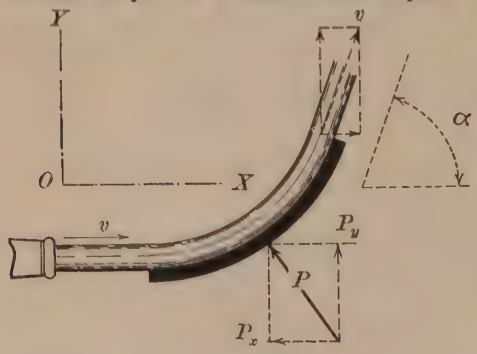

FIG. 128

Assuming the surface to be smooth, the jet will pass over it with practically undiminished velocity. The X component of P may be found by equating it to the change in momentum taking place between the points where the jet enters upon and leaves the surface, the momentum being calculated in a direction *parallel* to the X axis (in this case parallel to the original direction of the jet). The components of velocity at these two points being v and $v \cos \alpha$ respectively, we may write

$$P_x = M\,(v - v \cos \alpha)$$

or
$$P_x = Mv\,(1 - \cos \alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad (156)$$

Similarly,
$$P_y = Mv \sin \alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (157)$$

For the value of P we have

$$P = \sqrt{P_x^2 + P_y^2} = Mv\sqrt{2\,(1 - \cos \alpha)} \quad . \quad . \quad . \quad . \quad (158)$$

This value might have been obtained directly from a consideration of the vector change in velocity, or momentum, occurring between the two sections. The diagram (Fig. 129) of velocities shows v_1 and v_2 (alike in magnitude) as the initial and final velocities at the two sections, and the value of Δv must be the closing side of the triangle. By the law of cosines,

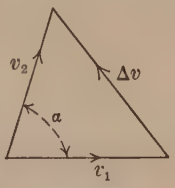

Fig. 129

$$\Delta v = \sqrt{v_1^2 + v_2^2 - 2\,v_1 v_2 \cos \alpha},$$

or

$$\Delta v = \sqrt{2\,v^2 - 2\,v^2 \cos \alpha}.$$

By equation 155,

$$P = Mv\sqrt{2\,(1 - \cos \alpha)}.$$

141. Special Cases.—Case I. Flat Plate Perpendicular to Jet. (Fig. 130). Here the angle through which the jet is deflected is 90 degrees and, by the application of (156) we find that

$$P = Mv. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (159)$$

It is not possible to use (158) as that equation presupposes that all the water is turned in the *same direction* through an angle α, in which case the force P would have both X and Y components as in Fig. 127. By reason of the fact that the stream is turned in all directions through 90 degrees, there is no resulting Y component and the value of the X components is the value of P. The result might be reached by noting that the stream's momentum in its initial direction is wholly destroyed and the force that caused it must be given by the relation

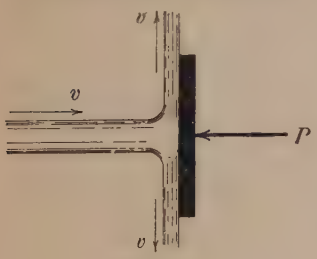

Fig. 130

$$P = Mv.$$

Case II. Jet Turned Through 180°.—If the surface be so formed as to cause the jet to be turned completely back upon itself (Fig. 131*a* and *b*), we have $\alpha = 180°$ and, by equation (156)

$$P = Mv\,[1 - (-1)] = 2\,Mv. \quad . \quad . \quad . \quad . \quad . \quad (160)$$

Again (156) is used instead of (158) for the same reason as given in Case I. The force exerted by the jet is thus twice as great as though directed normally against a flat plate. Applying the principle of momentum directly, we find that the total change in momen-

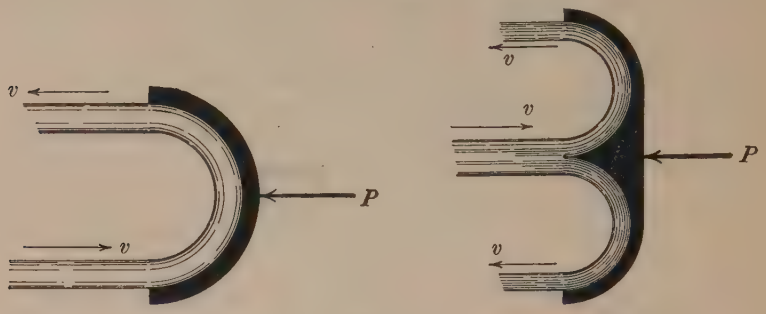

FIG. 131

tum in the initial direction of the jet is from Mv to $-Mv$ or $2Mv$. The force is therefore

$$P = 2Mv.$$

142. Dynamic Pressure in Pipe Bends.—(Fig. 132.) If a pipe or conduit in which water flows with a velocity, v, be bent on a curve through a deflection angle α, a dynamic pressure will exist on the bend as in the case of a jet deflected by a curved surface. The treatment of the two problems is identical, and for the resultant pressure we find

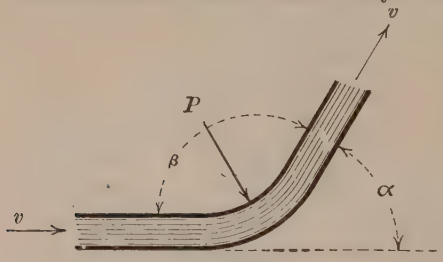

FIG. 132

$$P = Mv\sqrt{2(1 - \cos\alpha)}$$

as in Art. 139. The direction of the resultant P will bisect the angle β. Unless the pipe be rigid there is a tendency to disrupt it, and if it be not embedded in firm material it must be held in place by clamps, bands or other means unless the magnitude of P is negligible.

Example.—A 36-inch pipe, discharging 50 cu. ft. per second, contains a one-piece bend of 90°. Find the stress in two anchor rods placed radially at the ends of the bend.

$$v = 50 \div \frac{\pi \times 9}{4} = 7.1 \text{ ft. per second.}$$

By equation (121)

$$\text{Stress in one rod} = \frac{50 \times 62.5 \times 7.1}{32.2} (1 - \cos 90°) = 690 \text{ lb.}$$

143. Force Exerted by a Jet upon a Moving Curved Surface.— (Fig. 133.) Here the curved surface shown in Fig. 128, is assumed to have a motion in the same direction as the jet but a lesser velocity, u. The motion is also uniform, a suitable resistance R preventing any acceleration. At the point A, where the jet is tangent (preventing impact) to the surface, the velocity of the jet *relative to the surface* is $v - u$. If the surface be smooth and the

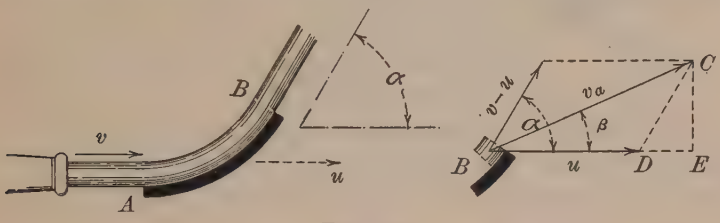

FIG. 133

vertical distance between A and B small, the relative velocity at B is also $v - u$. Therefore a particle arriving at B has not only a velocity of $v - u$ relative to the surface, but also a velocity of translation u. Its *absolute* velocity, v_a, is, therefore, represented in the second figure by the diagonal of the parallelogram of velocities. The component of v_a in a direction parallel to the initial direction of the jet is $v_a \cos \beta$, and the total change in momentum in this direction, effected while passing over the surface, is $M'(v - v_a \cos \beta)$.

The jet being horizontal, we have

$$\begin{aligned} H \text{ comp. of } P &= M'(v - v_a \cos \beta) \\ &= M'\{v - [u + (v - u) \cos \alpha]\} \\ &= M'(v - u)(1 - \cos \alpha) \quad . \quad . \quad . \quad . \quad . \quad (161) \end{aligned}$$

By a similar process of reasoning we find the value of the V component of P to be

$$V \text{ comp. of } P = M'(v - u) \sin \alpha. \quad . \quad . \quad . \quad . \quad (162)$$

Here M' is the mass of water which flows over the vane in one second of time and is *not* the same in value as M, which is the mass of water discharged each second by the jet. However, if we arrange a *series* of vanes, following one behind the other, so that, as each vane in succession cuts off the water from its predecessor, the water so cut off from the jet is left free to expend its kinetic energy on the first vane, then *all* the water in the jet may be utilized and M' be replaced by M in equations (161) and (162).

A common application of these principles is found in the construction and use of the so-called "Impulse wheels" for power development. Here the vanes or buckets are set at close intervals around the periphery of a wheel, and each receives in turn a pressure from a jet which is directed tangentially against the wheel. A section through one of the buckets, taken on a plane passing through the jet and perpendicular to a radius of the wheel, is sketched in Fig. 134. If the value of P is to be as large as possible for any given speed of the bucket, it will be a maximum when $\alpha = 180°$, in which case equation (161) becomes

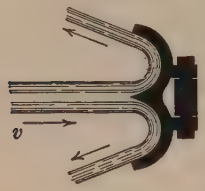

FIG. 134

$$P = 2\,M\,(v - u). \quad . \quad . \quad . \quad . \quad . \quad . \quad (163)$$

Equation (161) was used since it is the H component of pressure that turns the wheel and performs work. The V component is practically radial and performs no work. In practice, it is found impracticable to make $\alpha = 180°$ as the water discharge from each vane interferes with the vane immediately following. The usual angle is from 165° to 170°.

144. Power Exerted on Moving Vanes.—In Fig. 133 the only motion taking place was in a horizontal direction parallel to the jet. Therefore, the power (energy per second) to be derived from its motion will be the product of the H component of P and the distance u through which the component moves in one second.

$$\text{Power} = M'\,(v - u)\,(1 - \cos\alpha)\,u, \quad . \quad . \quad . \quad . \quad (164)$$

M' being equal to $\dfrac{wa\,(v - u)}{g}$. If we make u a variable in this

equation and equate $\frac{\delta \text{ (Power)}}{\delta u}$ to zero we obtain

$$u = \frac{v}{3}$$

as the value of u which will make the power a maximum for any given vane angle, α. The result, however, has no practical application since with only a single vane the vane would soon pass beyond the range of the jet.

With a *series* of vanes, or buckets, as arranged about the periphery of an impulse wheel, a practical machine is obtained in which M' assumes the value $M = \frac{wav}{g}$, as explained in the previous article, and the equating of $\frac{\delta \text{ (Power)}}{\delta u}$ to zero results in,

$$u = \frac{v}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (165)$$

as the peripheral speed of the buckets which gives a maximum power.

With these special values for M' and u substituted in (164) we obtain,

$$\text{Power} = \frac{M v^2}{4} (1 - \cos \alpha) \quad . \quad . \quad . \quad . \quad . \quad (166)$$

If α be made 180°,

$$\text{Power} = \frac{Mv^2}{2},$$

which is the expression for the kinetic energy in the jet. A wheel with buckets having $\alpha = 180°$ would thus have a hydraulic *efficiency* of 100 per cent., but the angle is an impracticable one as explained in the previous article.

145. Effect of Friction.—Equations 156 to 166 were derived without a consideration of the effect of friction on the force exerted. While there is no way in which this may be directly calculated, theory indicates the general effect and experiment has furnished certain numerical values. Friction between the deflecting surface and the stream results in a diminution of velocity at the second section, so that $v_2 < v_1$. Reference to Fig. 135*a* and *b*, where v_2 is drawn equal to v_1, shows that a decrease in v_2 will, for α less than

90°, result in a decrease in Δv up to a point where Δv in the diagram becomes perpendicular to v_2. Further decrease in v_2 will increase Δv. The total dynamic force P will change correspondingly. For angles of 90° or greater, changes in v_2 by friction produce corresponding (but not numerically equal) changes in Δv, hence in P. As for the effect of friction upon the component of P parallel to the initial direction of the jet, it will be noted (Fig. 135*a*) that the change in momentum in the same direction is indicated by Δv_x, which will increase as friction decreases v_2. The value of P_x, therefore, should increase with friction, and actual values slightly greater than those indicated by theory be obtained. For the special value of 90°, friction should have no effect on P_x. For

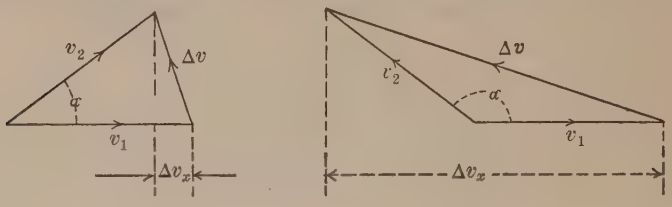

FIG. 135

angles greater than 90° (Fig. 135*b*), Δv_x will decrease with friction, as will P_x. This case is important as bearing upon the value of the turning force exerted by a stream on the buckets of an impulse wheel.

In some experiments by Gibson on flat plates against which a jet was directed normal to the plate, it was found that the ratio of actual force to theoretical force varied from about 0.83 to 0.96, depending on the velocity of the jet and the diameter of the plate. The diameter was an influencing factor because with small plates it was found that some of the water passed over the plate and escaped without having its direction changed by a full 90 degrees. The smaller the plate, the more noticeable was this effect.

Results obtained from tests made on impulse wheels where α had values between 165° and 175°, indicate that the ratio of the actual force to the theoretical force may be as high as 0.95 if the bucket be nicely designed and finished.

If a stream strikes a deflecting surface so that the angle of incidence be other than zero, friction by impact will be developed and also friction by subsequent eddying. The result is to further

diminish the value of v_2 with consequent change in the force developed.

146. Reaction of a Jet from an Orifice.—A jet escaping with a velocity v from an orifice in the vertical side of a reservoir, is composed of water particles which have had their velocity in a horizontal direction changed from zero to v. By the principles of Art. 138, it must have required a horizontal force, P, equal to Mv and acting in the direction of the jet to have produced this change. Reference to Fig. 136 will show that this force must be the resultant of all the horizontal forces acting on the mass, *cdmn*, in the direction of motion. Since the pressures of the walls, *cf* and *dg*, on the contained water are the only such forces, P must be the difference between these, or the pressure at the base of the jet. Considering the opposites of these two forces, we see that the water presses on the wall *cf* with a force greater, by the amount P, than it does on *dg*, so that the reservoir is subjected to an unbalanced force which would cause it to move in a direction opposite to that of the jet were it free to so do. This force may be regarded as the *Reaction* of the jet.

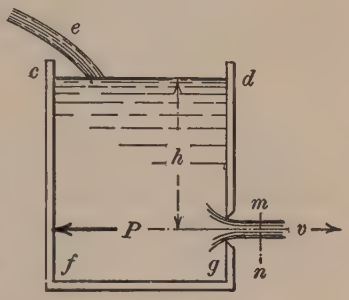

FIG. 136

Since $M = \frac{wav}{g}$, another expression for P would be

$$P = 2\,wa\frac{v^2}{2\,g} = 2\,awh.$$

From this it would appear that the force is twice the static pressure that would exist on a plug just filling the orifice. However, in deriving the expression no allowance was made for the contraction of the jet or for the effect of friction upon velocity. For a standard orifice, assuming $c_v = 0.98$, $c_c = 0.62$ and $c_d = 0.61$, we have

$$M = 0.62\,\frac{wav}{g} \text{ and } \frac{v^2}{2\,g} = (0.98)^2\,h,$$

since v is the actual velocity in the contracted section. Under these conditions

$$P = 2 \times 0.62\,wa \times (0.98)^2 h = 1.19\,awh.$$

Mr. Ewart of England made an actual measurement of P, using a standard orifice, and obtained a value of $1.14\,wh$.

147. Stress in Pipe Produced by Change in Section.—If a pipe be changed in section as shown in Fig. 137, the stream within exerts on the pipe a longitudinal thrust whose magnitude may be found as follows.

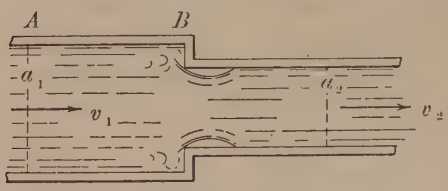

FIG. 137

Due to the change in section (it might have been gradual without changing the problem), the velocity is increased from v_1 to v_2, and the resultant force which caused the increase is, by Art. 138, equal to $M(v_2 - v_1)$. Three forces go to make up this resultant,—the axial component of the force which the pipe exerts upon the stream, and the pressures, a_1p_1 and a_2p_2, which act on the ends of the slug of water filling the pipe between the two sections shown. By equation (155)

$$a_1p_1 - a_2p_2 - P_x = M(v_2 - v_1),$$

or

$$Q\frac{p_1}{v_1} - Q\frac{p_2}{v_2} - P_x = \frac{wQ}{g}(v_2 - v_1),$$

and finally,

$$P_x = Q\left[\frac{p_1}{v_1} - \frac{p_2}{v_2} - w\left(\frac{v_2 - v_1}{g}\right)\right] \quad . \quad . \quad . \quad . \quad (167)$$

Example.—A pipe carrying 4 cu. ft. per second suffers a reduction in section from 2 sq. ft. to 1 sq. ft. If the pressure-head in the full section be 30 ft., find the dynamic force exerted upon the pipe.

From the data,

$$v_1 = 2 \text{ ft. per second.}$$

$$v_2 = 4 \text{ ft. per second.}$$

$$p_1 = 62.5 \times 30 = 1875 \text{ lb. per square foot.}$$

Between points located axially in the two pipes we may write

$$\frac{{v_1}^2}{2g} + \frac{p_1}{w} = \frac{{v_2}^2}{2g} + \frac{p_2}{w},$$

neglecting any loss of head. From this, p_2 is found to be 1865 lb. per square foot.

$$\therefore P = 4\left(\frac{1875}{2} - \frac{1865}{4} - \frac{62.5 \times 2}{32.2}\right) = 1872 \text{ lb.}$$

148. Pressure on Immersed Planes.—Case I. Normal to Current. If a plane surface or plate be immersed in a moving stream (Fig. 138) with its plane normal to the velocity of the stream, a dynamic pressure will be developed on the plate in much the same way as when a free jet is directed normally against a stationary flat plate (Art. 141). There is a great difference, however, in the two cases. In the latter, the derivation of formula (159) for the force developed was based on the assumption that the area of the jet's section was small when compared with the surface area of the plate, and that all the water in the jet was thus turned through an angle of 90° in passing over the plate. Had the plate been small in proportion to the section of the jet, much of the water in the outer portions of the jet would have been deflected through only a small angle and the value of the force would have been much less than as given by (159). This is approximately the condition of the immersed plate of Fig. 138. The water particles, which approach the plate at or near its center, have their velocity in a forward direction entirely suppressed and the pressure at this point is a maximum. Particles more remote from the center will be deflected through an angle varying with the distance of the particles from the center of the plate. Near the edges of the plate the deflection will be very small and the average deflection angle for all the particles will be much less than 90°. Formula (159) may be used as a rational basis for the value of the force developed between the stream and plate, but a coefficient must be introduced to allow for the difference, as just described, in the two cases. We may write,

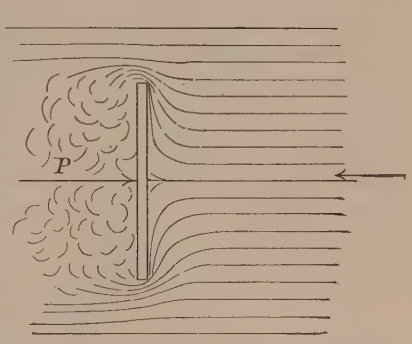

FIG. 138

$$P = k\,M\Delta\,v,$$

or

$$P = \frac{kwav^2}{g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (168)$$

and leave the determination of the value of k to experiment. The value of a is the area of the plate.

A factor which influences the magnitude of P is the formation of eddies at the edge of the plate. These move into the space just behind the plate and have the effect of reducing the pressure on its back face with a consequent increase in P.

Most of the experimental work has been conducted by moving the plates through still water and measuring the force or resistance encountered. This has generally been found to be less than the force necessary to hold a still plate against a moving stream and the difference is undoubtedly due to the larger eddy effect in the latter case.

Gibson gives values of k for some flat circular plates, moved through still water, as follows:

Diameter	k
1 inch	0.56
3 inch	0.65
6 inch	0.72

It is probable that a further increase in diameter would be attended by only a slight increase in k.

From the experiments of Dubuat and Duchemin, where a flat plate 1 foot square was moved through still water, a value for k of 0.72 was obtained.

A slight increase in the above values should be made for stationary plates.

Case II. Oblique to Current.—With the plate held obliquely to the current (Fig. 139) it has been found that the pressure against the plate is mainly normal to the plate, the tangential component being relatively small. Assuming that the value of the normal pressure is proportional to the momentum reduction in a normal direction, we have as the value of the latter,

$$\text{Normal vel. component} = v \sin \theta.$$

The area of the plate being a, the sectional area of the stream directly in front of the plate is $a \sin \theta$. Assuming the normal component of velocity of all particles to be reduced from $v \sin \theta$ to zero, we have

$$P_n = k' M \Delta v$$

or

$$P_n = k' \left[\frac{w (a \sin \theta) v}{g} \right] v \sin \theta,$$

or

$$P_n = k' \frac{wav^2}{g} \sin^2 \theta.$$

From this it would appear that the relation between the pressures

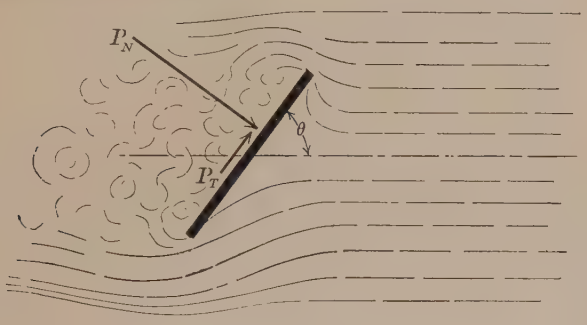

FIG. 139

on a plate held normal to a stream and the same plate held oblique to it may be written

$$P_n = \frac{k'}{k} (\sin^2 \theta) P,$$

P representing the pressure on the plate held normal to the stream. Experiments indicate that k and k' are not equal, k' varying with the angle of obliquity. More simply may we write

$$P_n = KP. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (169)$$

and depend upon experiment to indicate values of K. Again, as in the case of plates held normal to the stream, experimental work has been conducted by moving the plates through still water. Based upon the works of Stanton and Duchemin the following may be taken for values of K.

Angle θ	90°	80°	70°	60°	50°	40°	30°	20°	10°
K	1.00	1.00	1.00	0.98	0.95	0.90	0.85	0.60	0.30

The value of P must be obtained from equation (168) and for ordinary cases k may be assumed as 0.80.

149. Pressure on Immersed Bodies.—The scope of this particular subject is too great for inclusion in an elementary text, but a few facts are of sufficient importance to warrant their presentation here. Fig. 140 shows a prismatic shaped object held in a

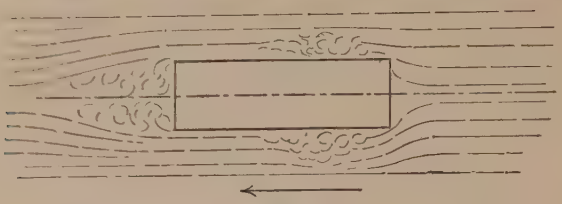

FIG. 140

moving stream or moved through still water. If the length of the prism be at least three times its face width, it has been found that the resistance developed between it and the water is less than for a flat plate of the size and shape of the front face. This is probably due to the part suppression of the eddies, formed at the face edges, as they move along the sides of the prism. This would result in less eddying at the rear face and a consequent decrease in the resistance developed (Art. 148). Thus, putting the resistance equal to $\frac{kwav^2}{g}$, experiments by Dubuat on a prism of the above proportions indicate a value for k of 0.66. If the front of the prism be tapered the value of k is reduced to about 0.40, and if the rear end be given a tapering stern the value of k may be cut to about 0.10. Both changes affect the amount of eddying produced and it will be noticed that the figures given indicate a finely

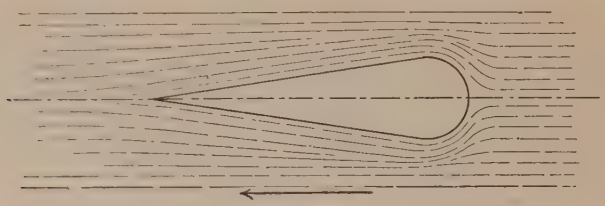

FIG. 141

tapered stern is of more effect than a tapered stem or bow. It is for this reason that much care is given the shaping of the stern of ships and the bodies of aeroplanes. Actual tests with models shaped as in Fig. 141 show that the resistance between the model

and the water is less when the water moves in the direction of the arrow than when the tapered end becomes the cutwater.

150. Water Hammer in Pipes.—It has long been a well-known fact that the sudden stopping of the flow of water in a pipe will cause a rise in pressure in that portion of the pipe lying between the source of flow and the gate or valve by which the flow is checked. In some cases the rise in pressure may be so large as to cause a bursting of the pipe, and it becomes a matter of importance that we determine the laws which govern the appearance of this phenomenon.

That we may better understand what actually takes place under such conditions, Fig. 142 has been drawn, showing a long pipe line of constant diameter leading from a reservoir of water, and provided at its further end with a quick-closing valve G.

Under normal conditions the flow through the pipe is steady, having a velocity v past all sections. If the gate be made to close

FIG. 142

instantaneously, the particles of water in immediate proximity to it will have their velocity at once reduced from v to zero. If the whole mass of water in the pipe were inelastic (rigid) and contained in pipe walls that were inelastic also, then all the particles of water would likewise be instantaneously brought to rest and the pressure against the gate and all through the pipe would be infinite. This follows from the relation

$$\text{Force} = \text{Mass} \times \text{Acceleration},$$

the acceleration in this case being

$$a = \frac{dv}{dt} = \frac{v - 0}{0} = \text{Infinity}.$$

But the pressure is not infinite, due to the fact that the water is elastic, therefore compressible, and the pipe walls are elastic also. To understand the effect which these conditions have, let us assume the water in the pipe to be divided into a great number of

thin laminæ, some of which are shown, greatly exaggerated in size, in Figure 142. When the gate closes, lamina 1 crowds up against it and is compressed by virtue of its own kinetic energy. As it is compressed, the ring of pipe wall surrounding it is distended by the increased pressure in the lamina. While the first lamina is being compressed and shortened, lamina 2 follows on behind with undiminished velocity until the compression of lamina 1 is complete. It then suffers retardation and compression, at the same time stretching the pipe wall around it. Other laminæ follow in succession, so that in a very short time conditions are as shown in Fig. 143. The pressure throughout the distended portion of the pipe has been increased at every point by the change

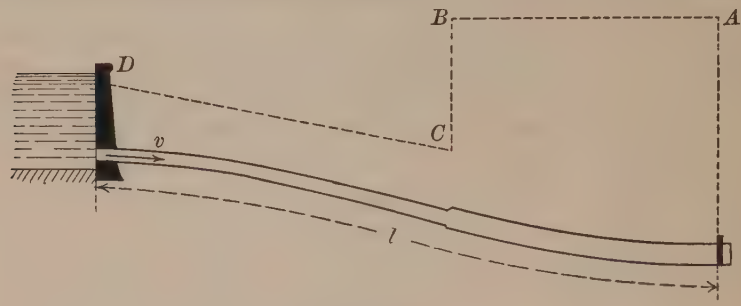

FIG. 143

of the kinetic energy of each lamina into pressure-energy. The laminæ being of equal mass, this rise in pressure is uniform at all points, and the pressure at any point in the distended portion will be the sum of the original pressure there during flow, the increment due to checked velocity, and the static pressure arising from the water now at rest in the distended portion. The hydraulic grade line is some line *ABCD*. In the *unstretched* portion of the pipe the velocity v is *still maintained*. When the last lamina at the entrance to the pipe is compressed, the pipe is filled with water under a pressure much in excess of that which previously existed. If the length of the pipe be l, and t seconds have elapsed between the time of closure of the valve and the compression of the last lamina, then a wave of pressure has swept up the pipe with a velocity $v_p = l \div t$. The numerical value of this velocity varies with the density of the water and the thickness and elasticity of the pipe walls.

With the last lamina brought to rest, the total kinetic energy, which the water had, has been transformed and stored up in the

elastic deformation of the water and pipe-walls. To more easily describe the continuance of the phenomena let us say that the attainment of this condition marks the end of the first period.

Second Period. This condition of inequipotential cannot be maintained, for as soon as the last lamina is compressed it will expand and the ring of pipe around it will contract, thus forcing the lamina out of the pipe into the reservoir, where it will resume its original condition of form and pressure. This follows with each lamina in succession until number 1 is reached and the whole pipe has been restored to its original size and the water in it to its original normal pressure. Each lamina will also have acquired its original velocity v, but the motion will be in the *opposite direction*, toward the reservoir.

Third Period. The kinetic energy of the water column, now moving toward the reservoir, will be expended in lowering the pressure in the laminæ (commencing with number 1 and continuing until the last lamina is reached) to a point *below normal*, and the water in the pipe will again come to rest.

Fourth Period. Water from the reservoir now enters the pipe. The nearest lamina resumes normal pressure and moves with its original velocity toward the gate. The other laminæ follow in succession, and finally all are moving with velocity v in the original direction and under normal pressure. Conditions are now the same as at the beginning of operations. We have, therefore, completed a cycle of four movements, each one extending through the whole length of pipe. Two successive movements may be termed a "round trip" of the pressure wave. Other cycles follow and were the whole operation unaccompanied by friction, the phenomena would continue indefinitely. Friction in the water and the walls of the pipe causes each cycle to take place with diminished energy and the pressure waves gradually die out.

We shall now proceed to determine the value of p, the pressure in excess of that which existed at any point in the pipe during flow.

In Fig. 144, $ABCD$ represents a prism of water caught in the end of the pipe by the instantaneous closing of the gate G. It contains many of the laminæ mentioned above, and by their successive compression the prism is shortened by the amount ds. Its final length becomes ds', and by reason of the stretching of the pipe, its radius is increased to $(r + \Delta r)$. For the change in volume we may write

$$\Delta V = A \cdot ds - 2\,\pi r \cdot \Delta r \cdot ds',$$

where A is the cross-sectional area of the pipe before distention, and $2\,\pi r \times \Delta r$ represents approximately the annular area added by the distention. The modulus of elasticity E is therefore (Art. 2),

$$E = \frac{pA(ds + ds')}{A \cdot ds - 2\,\pi r \cdot \Delta r \cdot ds'}. \quad . \quad . \quad . \quad (170)$$

Here p represents the intensity of the excess pressure arising from the compression of the laminæ. The stress produced in the pipe-wall by this pressure is (Art. 23) $p' = pr \div T$. If under this stress

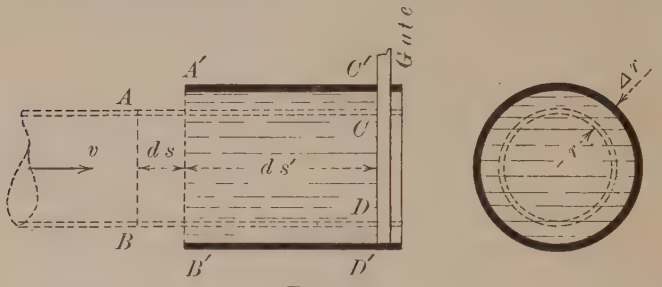

Fig. 144

the length of the circumference is stretched a distance Δl, we have for E', the elastic modulus of the pipe-wall material,

$$E' = \frac{p'}{\Delta l \div 2\,\pi r} = \frac{pr \cdot 2\,\pi r}{\Delta l \cdot T}. \quad . \quad . \quad . \quad . \quad (171)$$

Since a circumference increases with the radius, we may write

$$\Delta l : 2\,\pi r = \Delta r : r,$$

or

$$\Delta l = 2\,\pi \Delta r.$$

Equation 171 may then be written

$$\Delta r = \frac{pr^2}{TE'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (172)$$

Let us now return to a consideration of the prism $ABCD$. Each of its laminæ is brought to rest from a velocity v by a pressure on its forward face varying in intensity from zero to p. If, now, we imagine the first lamina to have reached its maximum pressure, then we may think of the entire mass $ABCD$ behind it as being brought to rest by the force pA acting on its end face. In coming

to rest the center of gravity of the mass will have moved through a distance $ds \div 2$. Now the space traversed by a body while being brought to rest from an initial velocity v under the action of a constant force is

$$s = v^2 \div 2 \times \text{Acceleration.}$$

The acceleration (negative) of the mass $ABCD$ being

$$a = \frac{\text{Force}}{\text{Mass}} = \frac{p \cdot A \cdot g}{A(ds + ds')w},$$

we may substitute in the above formula, obtaining

$$\frac{ds}{2} = \frac{1}{2} \cdot \frac{v^2 A(ds + ds')w}{pAg},$$

or more simply, since $\frac{ds}{dt} = v$,

$$p \cdot g \cdot dt = (ds + ds')wv. \quad . \quad . \quad . \quad . \quad . \quad (173)$$

The quantity ds being very small compared with ds' we may write without appreciable error,

$$p = \frac{wv}{g} \times \frac{ds'}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (174)$$

But $ds' \div dt$ represents the velocity with which the pressure wave moves along the pipe, and its value may be determined as follows: If we substitute in (170) the value of Δr as given in (172) and for A its value πr^2, we may combine (170) and (173), eliminating $(ds + ds')$.

There results,

$$\frac{ds}{dt} - \frac{2rp}{TE'} \times \frac{ds'}{dt} = \frac{p^2 g}{Evw}. \quad . \quad . \quad . \quad . \quad . \quad (175)$$

But $ds \div dt$ we know to be v, and for the value of p we may substitute from (174).

$$\therefore \text{Vel. of pressure wave} = \frac{ds'}{dt} = \sqrt{\frac{g}{w} \cdot \frac{EE'T}{TE' + 2rE}}. \quad . \quad (176)$$

This value of $ds' \div dt$ substituted in (174) gives finally for the excess pressure,

$$p = v\sqrt{\frac{wEE'T}{g(TE' + 2rE)}}. \quad . \quad . \quad . \quad . \quad . \quad (177)$$

It is of interest to note that this formula gives p as independent of the *length of pipe* and consequently of the *mass of water flowing in it.* It is to be remembered, however, that the formula is based on the assumption that the closure of the gate is *instantaneous.*

In 1897–98 Professor Joukovsky of Moscow, Russia, carried out an elaborate and most carefully conducted series of experiments on water hammer, using pipes of 2, 4, 6, and 24 inches diameter. Their lengths were 2494, 1050, 1066, and 7007 ft. respectively. His experimental results substantiated the truth of the above derived formulas and showed the effect which slow closure of the gate has on the value of p. He found that, if the time of closure is not greater than $2l \div v_p$, which is the time required for the pressure wave to make a "round trip" from the gate to the reservoir and back, then the full force of the maximum pressure p will be felt in the pipe. If a longer time t' be taken in moving the gate, then the pressure is reduced in intensity according to the proportion

$$\frac{p}{p_{max}} = \frac{t_r}{t'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (178)$$

A very well written account of Joukovsky's experiments and deductions may be found in the 1904 Proceedings of the American Waterworks Association, p. 335. Under the head of Summary and Conclusions the author states,—

"The simplest method of protecting water pipes from water hammer is found in the use of slow-closing gates. The duration of closure should be proportional to the length of the pipe-line. Air chambers, of adequate size, placed near the valves and gates, eliminate almost entirely the hydraulic shock, and do not allow the pressure wave to pass through them; but they must be very large, and it is difficult to keep them supplied with air. Safety valves allow to pass through them pressure waves of only such intensity as corresponds to the elasticity of the springs of the safety valves."

In the analytical discussion presented in this article the author has followed, with few exceptions, the method of deduction given by Professor Church in his "Hydraulic Motors," and its use here is with that writer's kind permission.

PROBLEMS

1. A jet of water, 2 inches in diameter, is directed against a flat plate held normal to the stream's axis. Compute the pressure exerted on the plate by the jet when the velocity of the latter is 90 ft. per second.

2. A jet of water 1 in. in diameter exerts a pressure of 150 lb. against a flat plate held normal to the stream's axis. Compute the rate of discharge. *Ans.* 0.65 cu. ft. per second.

3. A jet from a 2-inch orifice is directed normally against a flat plate. What pressure will it exert on the plate when $Q=1$ cu. ft. per second? Assume $c_c=0.62$. *Ans.* 144 lb.

4. A jet 2 in. in diameter, having a velocity of 140 ft. per second, is deflected by a curved surface through an angle of 45 degrees. Compute the value of the components of the pressure developed which are perpendicular and parallel respectively to the initial direction of the jet. Also compute the parallel component for a deflection of 90 degrees and 180 degrees.

Ans. (*a*) 586 lb.; 243 lb.
(*b*) 829 lb.
(*c*) 1658 lb.

5. For the data given in Problem 4, compute the value of the required components, assuming that frictional resistance causes the velocity of the water to be reduced, in passing over the surface, from 140 to 120 ft. per second. Compare results with those previously obtained.

6. In approaching a bridge, a water main, 2 ft. in diameter, curves, from a horizontal position, upward through 45 degrees. What vertical component of dynamic pressure is developed in the bend under a velocity of 6 ft. per second? *Ans.* 155 lb.

7. A 9-ft. wood-stave pipe, carrying water with a mean velocity of 10 ft. per second, contains a 90° bend made of riveted steel pipe. To prevent rupture, the bend is buried in a block of concrete weighing 135 lb. per cu. ft. How many cubic feet will be required if the coefficient of friction of the concrete on its foundation be assumed equal to 0.30?

8. A 36-inch pipe is changed by gradual reduction to a 24-inch diameter. Before entering the reducer the water has a velocity of 7 ft. per second and a mean pressure of 60 lb. per square inch. A loss of 0.20 ft. head is experienced in going through the reducer. What axial thrust on the pipeline does the water develop as it passes the reducer?

Ans. 33,800 lb.

9. Compute the force exerted (in an axial direction) on a pipe by water which is flowing through it with a velocity of 8 ft. per second. The diameter is 1 ft., length 1000 ft., and f may be assumed as 0.02.

Ans. 973 lb.

10. A 2-inch nozzle is attached to a 12-inch pipe by flange bolts. What will be the total stress in the bolts when the pressure at the base of the nozzle is 75 lb. per square inch? Assume $c_d = c_v = 0.96$.

Ans. 8050 lb.

11. A 2-inch jet from a nozzle is deflected through an angle of 175 degrees by striking against a curved vane which moves in the same direction as the jet and with a velocity of 100 ft. per second. What pressure will be on the vane when the nozzle is discharging 6 cu. ft. per second? What if the vane were stationary?

Ans. (*a*) 2590 lb.; (*b*) 6390 lb.

12. Assume the data of the previous problem except that the velocity of the jet relative to the vane's surface is gradually reduced by frictional resistance until at the point of leaving the vane it is

(*a*) 150 ft. per second with the vane moving.

(*b*) 200 ft. per second with the vane stationary. Compute the pressure asked for.

13. A primitive form of water turbine consisted of flat plates set radially around the periphery of a wheel. Power was derived from the action of a jet which was directed perpendicularly against the surface of the plates. Assuming a 2-inch jet moving with a velocity of 120 ft. per second, what maximum theoretical amount of power could be derived from such a wheel? What per cent of the energy of the jet would be utilized?

14. The buckets of an impulse wheel receive water from a 2-inch jet which has a velocity of 60 ft. per second. If the linear speed of the buckets be 40 ft. per second, and the bucket angle, α, be 170°, find the turning force applied to the wheel, assuming all water coming from the jet to be utilized. At what speed should the buckets be made to move in order that the power output of the wheel should be a maximum? What would the power be?

Ans. (*a*) 101 lb.; (*b*) 30 ft. per second; (*c*) 4540 ft. lb.

15. A 9-inch jet having a velocity of 280 ft. per second furnishes power to an impulse wheel whose buckets are curved through 165°. If 10 per cent of the *relative* velocity-head be lost in passing over the buckets, what power may be derived from the water when the linear velocity of the buckets is 125 ft. per second?

16. A ship moves forward under the driving force obtained by steadily pumping a stream of water, 1 ft. in diameter, having a velocity relative to the ship of 100 ft. per second, directly astern. Compute the driving force.

Ans. 15,250 lb.

17. A fire boat is equipped with a 4-inch nozzle ($c_v = c_d = 0.98$) supplied by a vertical pipe from the pumps below. The nozzle is jointed to the pipe and free to move in a vertical or horizontal plane. What driving force would the boat experience if the nozzle was directed horizontally astern while discharging 4000 gals. per minute?

18. A flat plate 20 ft. square is immersed in a stream of water running with a velocity of 10 miles per hour. Compute the approximate pressure developed against it when held normal to the current. What if held at 45° with the current? At 60°?

19. A rectangular caisson is to be floated in a stream and held in place by anchorages. The face of the caisson is 20 ft. wide and extends 20 ft. below the water surface. Its length (parallel to stream's axis) is 40 ft. If the velocity of the stream attains a maximum of 10 miles per hour, what horizontal component parallel to the stream's direction must be withstood by the anchorages?

20. Assuming that air in motion follows the same law as water, what pressure would be developed by a gale of 70 miles per hour against a vertical bill board 40 ft. long and 10 ft. high? Assume air to weigh 0.08 lb. per cu. ft.

21. If the velocity of flow in a 2-foot cast-iron pipe ($E = 12{,}000{,}000$) be changed in 0.25 sec. from 2 ft. per second to 0 by closing a valve 1000 ft. from a reservoir, what probable increase in pressure due to water hammer will be obtained close to the valve? The pipe wall is $\frac{3}{4}$ in. thick.

Ans. 94.2 lb. per square inch.

22. A 6-ft. pipe conducting water to a number of turbines is made from steel plate 0.25 in. thick. When the velocity of flow is 8 ft. per second, a quick closing gate operates to stop the flow. What excess pressure is developed near the gate due to water hammer? What maximum time may be taken in closing the gate without diminishing this pressure if pipe be 15,200 ft. long. Estimate the probable time that should be taken if the rise in pressure is not to exceed 100 lb. per square inch.

Ans. (*a*) 258 lb. per square inch.
(*b*) 12.8 sec.
(*c*) 33 sec.

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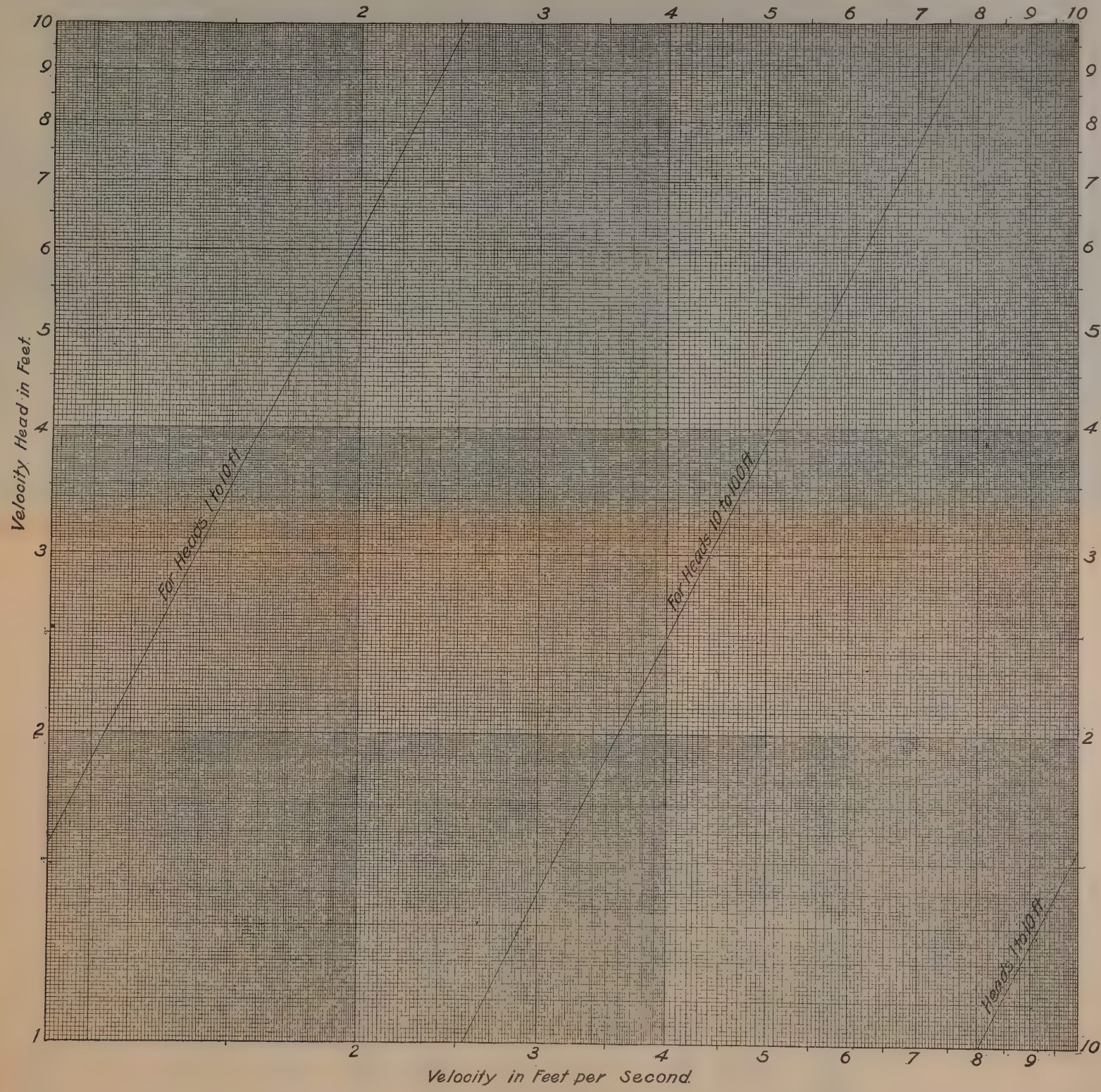

CONVERSION DIAGRAM

VELOCITY *v.* VELOCITY HEAD

$$\left(\frac{v^2}{2g} = h\right)$$

NOTE. — The marginal figures represent the first significant figure only, and the position of the decimal point must be determined by inspection.

Degrees	SINES							Cosines
	0′	10′	20′	30′	40′	50′	60′	
0	0.00000	0.00291	0.00582	0.00873	0.01164	0.01454	0.01745	89
1	0.01745	0.02036	0.02327	0.02618	0.02908	0.03199	0.03490	88
2	0.03490	0.03781	0.04071	0.04362	0.04653	0.04943	0.05234	87
3	0.05234	0.05524	0.05814	0.06105	0.06395	0.06685	0.06976	86
4	0.06976	0.07266	0.07556	0.07846	0.08136	0.08426	0.08716	85
5	0.08716	0.09005	0.09295	0.09585	0.09874	0.10164	0.10453	84
6	0.10453	0.10742	0.11031	0.11320	0.11609	0.11898	0.12187	83
7	0.12187	0.12476	0.12764	0.13053	0.13341	0.13629	0.13917	82
8	0.13917	0.14205	0.14493	0.14781	0.15069	0.15356	0.15643	81
9	0.15643	0.15931	0.16218	0.16505	0.16792	0.17078	0.17365	80
10	0.17365	0.17651	0.17937	0.18224	0.18509	0.18795	0.19081	79
11	0.19081	0.19366	0.19652	0.19937	0.20222	0.20507	0.20791	78
12	0.20791	0.21076	0.21360	0.21644	0.21928	0.22212	0.22495	77
13	0.22495	0.22778	0.23062	0.23345	0.23627	0.23910	0.24192	76
14	0.24192	0.24474	0.24756	0.25038	0.25320	0.25601	0.25882	75
15	0.25882	0.26163	0.26443	0.26724	0.27004	0.27284	0.27564	74
16	0.27564	0.27843	0.28123	0.28402	0.28680	0.28959	0.29237	73
17	0.29237	0.29515	0.29793	0.30071	0.30348	0.30625	0.30902	72
18	0.30902	0.31178	0.31454	0.31730	0.32006	0.32282	0.32557	71
19	0.32557	0.32832	0.33106	0.33381	0.33655	0.33929	0.34202	70
20	0.34202	0.34475	0.34748	0.35021	0.35293	0.35565	0.35837	69
21	0.35837	0.36108	0.36379	0.36650	0.36921	0.37191	0.37461	68
22	0.37461	0.37730	0.37999	0.38268	0.38537	0.38805	0.39073	67
23	0.39073	0.39341	0.39608	0.39875	0.40142	0.40408	0.40674	66
24	0.40674	0.40939	0.41204	0.41469	0.41734	0.41998	0.42262	65
25	0.42262	0.42525	0.42788	0.43051	0.43313	0.43575	0.43837	64
26	0.43837	0.44098	0.44359	0.44620	0.44880	0.45140	0.45399	63
27	0.45399	0.45658	0.45917	0.46175	0.46433	0.46690	0.46947	62
28	0.46947	0.47204	0.47460	0.47716	0.47971	0.48226	0.48481	61
29	0.48481	0.48735	0.48989	0.49242	0.49495	0.49748	0.50000	60
30	0.50000	0.50252	0.50503	0.50754	0.51004	0.51254	0.51504	59
31	0.51504	0.51753	0.52002	0.52250	0.52498	0.52745	0.52992	58
32	0.52992	0.53238	0.53484	0.53730	0.53975	0.54220	0.54464	57
33	0.54464	0.54708	0.54951	0.55194	0.55436	0.55678	0.55919	56
34	0.55919	0.56160	0.56401	0.56641	0.56880	0.57119	0.57358	55
35	0.57358	0.57596	0.57833	0.58070	0.58307	0.58543	0.58779	54
36	0.58779	0.59014	0.59248	0.59482	0.59716	0.59949	0.60182	53
37	0.60182	0.60414	0.60645	0.60876	0.61107	0.61337	0.61566	52
38	0.61566	0.61795	0.62024	0.62251	0.62479	0.62706	0.62932	51
39	0.62932	0.63158	0.63383	0.63608	0.63832	0.64056	0.64279	50
40	0.64279	0.64501	0.64723	0.64945	0.65166	0.65386	0.65606	49
41	0.65606	0.65825	0.66044	0.66262	0.66480	0.66697	0.66913	48
42	0.66913	0.67129	0.67344	0.67559	0.67773	0.67987	0.68200	47
43	0.68200	0.68412	0.68624	0.68835	0.69046	0.69256	0.69466	46
44	0.69466	0.69675	0.69883	0.70091	0.70298	0.70505	0.70711	45
Sines	60′	50′	40′	30′	20′	10′	0′	Degrees
	COSINES							

Degrees	COSINES							Sines
	0′	10′	20′	30′	40′	50′	60′	
0	1.00000	1.00000	0.99998	0.99996	0.99993	0.99989	0.99985	89
1	0.99985	0.99979	0.99973	0.99966	0.99958	0.99949	0.99939	88
2	0.99939	0.99929	0.99917	0.99905	0.99892	0.99878	0.99863	87
3	0.99863	0.99847	0.99831	0.99813	0.99795	0.99776	0.99756	86
4	0.99756	0.99736	0.99714	0.99692	0.99668	0.99644	0.99619	85
5	0.99619	0.99594	0.99567	0.99540	0.99511	0.99482	0.99452	84
6	0.99452	0.99421	0.99390	0.99357	0.99324	0.99290	0.99255	83
7	0.99255	0.99219	0.99182	0.99144	0.99106	0.99067	0.99027	82
8	0.99027	0.98986	0.98944	0.98902	0.98858	0.98814	0.98769	81
9	0.98769	0.98723	0.98676	0.98629	0.98580	0.98531	0.98481	80
10	0.98481	0.98430	0.98378	0.98325	0.98272	0.98218	0.98163	79
11	0.98163	0.98107	0.98050	0.97992	0.97934	0.97875	0.97815	78
12	0.97815	0.97754	0.97692	0.97630	0.97566	0.97502	0.97437	77
13	0.97437	0.97371	0.97304	0.97237	0.97169	0.97100	0.97030	76
14	0.97030	0.96959	0.96887	0.96815	0.96742	0.96667	0.96593	75
15	0.96593	0.96517	0.96440	0.96363	0.96285	0.96206	0.96126	74
16	0.96126	0.96046	0.95964	0.95882	0.95799	0.95715	0.95630	73
17	0.95630	0.95545	0.95459	0.95372	0.95284	0.95195	0.95106	72
18	0.95106	0.95015	0.94924	0.94832	0.94740	0.94646	0.94552	71
19	0.94552	0.94457	0.94361	0.94264	0.94167	0.94068	0.93969	70
20	0.93969	0.93869	0.93769	0.93667	0.93565	0.93462	0.93358	69
21	0.93358	0.93253	0.93148	0.93042	0.92935	0.92827	0.92718	68
22	0.92718	0.92609	0.92499	0.92388	0.92276	0.92164	0.92050	67
23	0.92050	0.91936	0.91822	0.91706	0.91590	0.91472	0.91355	66
24	0.91355	0.91236	0.91116	0.90996	0.90875	0.90753	0.90631	65
25	0.90631	0.90507	0.90383	0.90259	0.90133	0.90007	0.89879	64
26	0.89879	0.89752	0.89623	0.89493	0.89363	0.89232	0.89101	63
27	0.89101	0.88968	0.88835	0.88701	0.88566	0.88431	0.88295	62
28	0.88295	0.88158	0.88020	0.87882	0.87743	0.87603	0.87462	61
29	0.87462	0.87321	0.87178	0.87036	0.86892	0.86748	0.86603	60
30	0.86603	0.86457	0.86310	0.86163	0.86015	0.85866	0.85717	59
31	0.85717	0.85567	0.85416	0.85264	0.85112	0.84959	0.84805	58
32	0.84805	0.84650	0.84495	0.84339	0.84182	0.84025	0.83867	57
33	0.83867	0.83708	0.83549	0.83389	0.83228	0.83066	0.82904	56
34	0.82904	0.82741	0.82577	0.82413	0.82248	0.82082	0.81915	55
35	0.81915	0.81748	0.81580	0.81412	0.81242	0.81072	0.80902	54
36	0.80902	0.80730	0.80558	0.80386	0.80212	0.80038	0.79864	53
37	0.79864	0.79688	0.79512	0.79335	0.79158	0.78980	0.78801	52
38	0.78801	0.78622	0.78442	0.78261	0.78079	0.77897	0.77715	51
39	0.77715	0.77531	0.77347	0.77162	0.76977	0.76791	0.76604	50
40	0.76604	0.76417	0.76229	0.76041	0.75851	0.75661	0.75471	49
41	0.75471	0.75280	0.75088	0.74896	0.74703	0.74509	0.74314	48
42	0.74314	0.74120	0.73924	0.73728	0.73531	0.73333	0.73135	47
43	0.73135	0.72937	0.72737	0.72537	0.72337	0.72136	0.71934	46
44	0.71934	0.71732	0.71529	0.71325	0.71121	0.70916	0.70711	45
Cosines	60′	50′	40′	30′	20′	10′	0′	Degrees
	SINES							

Degrees	TANGENTS							Cotangents
	0′	10′	20′	30′	40′	50′	60′	
0	0.00000	0.00291	0.00582	0.00873	0.01164	0.01455	0.01746	89
1	0.01746	0.02036	0.02328	0.02619	0.02910	0.03201	0.03492	88
2	0.03492	0.03783	0.04075	0.04366	0.04658	0.04949	0.05241	87
3	0.05241	0.05533	0.05824	0.06116	0.06408	0.06700	0.06993	86
4	0.06993	0.07285	0.07578	0.07870	0.08163	0.08456	0.08749	85
5	0.08749	0.09042	0.09335	0.09629	0.09923	0.10216	0.10510	84
6	0.10510	0.10805	0.11099	0.11394	0.11688	0.11983	0.12278	83
7	0.12278	0.12574	0.12869	0.13165	0.13461	0.13758	0.14054	82
8	0.14054	0.14351	0.14648	0.14945	0.15243	0.15540	0.15838	81
9	0.15838	0.16137	0.16435	0.16734	0.17033	0.17333	0.17633	80
10	0.17633	0.17933	0.18233	0.18534	0.18835	0.19136	0.19438	79
11	0.19438	0.19740	0.20042	0.20345	0.20648	0.20952	0.21256	78
12	0.21256	0.21560	0.21864	0.22169	0.22475	0.22781	0.23087	77
13	0.23087	0.23393	0.23700	0.24008	0.24316	0.24624	0.24933	76
14	0.24933	0.25242	0.25552	0.25862	0.26172	0.26483	0.26795	75
15	0.26795	0.27107	0.27419	0.27732	0.28046	0.28360	0.28675	74
16	0.28675	0.28990	0.29305	0.29621	0.29938	0.30255	0.30573	73
17	0.30573	0.30891	0.31210	0.31530	0.31850	0.32171	0.32492	72
18	0.32492	0.32814	0.33136	0.33460	0.33783	0.34108	0.34433	71
19	0.34433	0.34758	0.35085	0.35412	0.35740	0.36068	0.36397	70
20	0.36397	0.36727	0.37057	0.37388	0.37720	0.38053	0.38386	69
21	0.38386	0.38721	0.39055	0.39391	0.39727	0.40065	0.40403	68
22	0.40403	0.40741	0.41081	0.41421	0.41763	0.42105	0.42447	67
23	0.42447	0.42791	0.43136	0.43481	0.43828	0.44175	0.44523	66
24	0.44523	0.44872	0.45222	0.45573	0.45924	0.46277	0.46631	65
25	0.46631	0.46985	0.47341	0.47698	0.48055	0.48414	0.48773	64
26	0.48773	0.49134	0.49495	0.49858	0.50222	0.50587	0.50953	63
27	0.50953	0.51320	0.51688	0.52057	0.52427	0.52798	0.53171	62
28	0.53171	0.53545	0.53920	0.54296	0.54674	0.55051	0.55431	61
29	0.55431	0.55812	0.56194	0.56577	0.56962	0.57348	0.57735	60
30	0.57735	0.58124	0.58513	0.58905	0.59297	0.59691	0.60086	59
31	0.60086	0.60483	0.60881	0.61280	0.61681	0.62083	0.62487	58
32	0.62487	0.62892	0.63299	0.63707	0.64117	0.64528	0.64941	57
33	0.64941	0.65355	0.65771	0.66189	0.66608	0.67028	0.67451	56
34	0.67451	0.67875	0.68301	0.68728	0.69157	0.69588	0.70021	55
35	0.70021	0.70455	0.70891	0.71329	0.71769	0.72211	0.72654	54
36	0.72654	0.73100	0.73547	0.73996	0.74447	0.74900	0.75355	53
37	0.75355	0.75812	0.76272	0.76733	0.77196	0.77661	0.78129	52
38	0.78129	0.78598	0.79070	0.79544	0.80020	0.80498	0.80978	51
39	0.80978	0.81461	0.81946	0.82434	0.82923	0.83415	0.83910	50
40	0.83910	0.84407	0.84906	0.85408	0.85912	0.86419	0.86929	49
41	0.86929	0.87441	0.87955	0.88473	0.88992	0.89515	0.90040	48
42	0.90040	0.90569	0.91099	0.91633	0.92170	0.92709	0.93252	47
43	0.93252	0.93797	0.94345	0.94896	0.95451	0.96008	0.96569	46
44	0.96569	0.97133	0.97700	0.98270	0.98843	0.99420	1.00000	45
Tangents	60′	50′	40′	30′	20′	10′	0′	Degrees
	COTANGENTS							

Degrees	COTANGENTS							Cotangents
	0′	10′	20′	30′	40′	50′	° 60′	
0	∞	343.77371	171.88540	114.58865	85.93979	68.75009	57.28996	89
1	57.28996	49.10388	42.96408	38.18846	34.36777	31.24158	28.63625	88
2	28.63625	26.43160	24.54176	22.90377	21.47040	20.20555	19.08114	87
3	19.08114	18.07498	17.16934	16.34986	15.60478	14.92442	14.30067	86
4	14.30067	13.72674	13.19688	12.70621	12.25051	11.82617	11.43005	85
5	11.43005	11,05943	10.71191	10.38540	10.07803	9.78817	9.51436	84
6	9.51436	9.25530	9.00983	8.77689	8.55555	8.34496	8.14435	83
7	8.14435	7.95302	7.77035	7.59575	7.42871	7.26873	7.11537	82
8	7.11537	6.96823	6.82694	6.69116	6.56055	6.43484	6.31375	81
9	6.31375	6.19703	6.08444	5.97576	5.87080	5.76937	5.67128	80
10	5.67128	5.57638	5.48451	5.39552	5.30928	5.22566	5.14455	79
11	5.14455	5.06584	4.98940	4.91516	4.84300	4.77286	4.70463	78
12	4.70463	4.63825	4.57363	4.51071	4.44942	4.38969	4.33148	77
13	4.33148	4.27471	4.21933	4.16530	4.11256	4.06107	4.01078	76
14	4.01078	3.96165	3.91364	3.86671	3.82083	3.77595	3.73205	75
15	3.73205	3.68909	3.64705	3.60588	3.56557	3.52609	3.48741	74
16	3.48741	3.44951	3.41236	3.37594	3.34023	3.30521	3.27085	73
17	3.27085	3.23714	3.20406	3.17159	3.13972	3.10842	3.07768	72
18	3.07768	3.04749	3.01783	2.98869	2.96004	2.93189	2.90421	71
19	2.90421	2.87700	2.85023	2.82391	2.79802	2.77254	2.74748	70
20	2.74748	2.72281	2.69853	2.67462	2.65109	2.62791	2.60509	69
21	2.60509	2.58261	2.56046	2.53865	2.51715	2.49597	2.47509	68
22	2.47509	2.45451	2.43422	2.41421	2.39449	2.37504	2.35585	67
23	2.35585	2.33693	2.31826	2.29984	2.28167	2.26374	2.24604	66
24	2.24604	2.22857	2.21132	2.19430	2.17749	2.16090	2.14451	65
25	2.14451	2.12832	2.11233	2.09654	2.08094	2.06553	2.05030	64
26	2.05030	2.03526	2.02039	2.00569	1.99116	1.97680	1.96261	63
27	1.96261	1.94858	1.93470	1.92098	1.90741	1.89400	1.88073	62
28	1.88073	1.86760	1.85462	1.84177	1.82907	1.81649	1.80405	61
29	1.80405	1.79174	1.77955	1.76749	1.75556	1.74375	1.73205	60
30	1.73205	1.72047	1.70901	1.69766	1.68643	1.67530	1.66428	59
31	1.66428	1.65337	1.64256	1.63185	1.62125	1.61074	1.60033	58
32	1.60033	1.59002	1.57981	1.56969	1.55966	1.54972	1.53987	57
33	1.53987	1.53010	1.52043	1.51084	1.50133	1.49190	1.48256	56
34	1.48256	1.47330	1.46411	1.45501	1.44598	1.43703	1.42815	55
35	1.42815	1.41934	1.41061	1.40195	1.39336	1.38484	1.37638	54
36	1.37638	1.36800	1.35968	1.35142	1.34323	1.33511	1.32704	53
37	1.32704	1.31904	1.31110	1.30323	1.29541	1.28764	1.27994	52
38	1.27994	1.27230	1.26471	1.25717	1.24969	1.24227	1.23490	51
39	1.23490	1.22758	1.22031	1.21310	1.20593	1.19882	1.19175	50
40	1.19175	1.18474	1.17777	1.17085	1.16398	1.15715	1.15037	49
41	1.15037	1.14363	1.13694	1.13029	1.12369	1.11713	1.11061	48
42	1.11061	1.10414	1.09770	1.09131	1.08496	1.07864	1.07237	47
43	1.07237	1.06613	1.05994	1.05378	1.04766	1.04158	1.03553	46
44	1.03553	1.02952	1.02355	1.01761	1.01170	1.00583	1.00000	45
Cotangents	60′	50′	40′	30′	20′	10′	0′	Degrees
	TANGENTS							

Diameter in Inches	Area		Diameter in Inches	Area		Diameter in Inches	Area	
	Square Inches	Square Feet		Square Inches	Square Feet		Square Inches	Square Feet
1.0	0.7854	.005454	**4.0**	12.5664	.087266	**7.0**	38.4845	.267254
.1	0.9503	.006599	.1	13.2025	.091684	.1	39.5919	.274944
.2	1.1310	.007854	.2	13.8544	.096211	.2	40.7150	.282743
.25	1.2272	.008522	.25	14.1863	.098516	.25	41.2825	.286684
.3	1.3273	.009218	.3	14.5220	.100847	.3	41.8539	.290652
.4	1.5394	.010690	.4	15.2053	.105592	.4	43.0084	.298669
.5	1.7671	.012272	.5	15.9043	.110446	.5	44.1786	.306796
.6	2.0106	.013963	.6	16.6190	.115410	.6	45.3646	.315032
.7	2.2698	.015763	.7	17.3494	.120482	.7	46.5663	.323377
.75	2.4053	.016703	.75	17.7205	.123059	.75	47.1730	.327590
.8	2.5447	.017671	.8	18.0956	.125664	.8	47.7836	.331831
.9	2.8353	.019689	.9	18.8574	.130954	.9	49.0167	.340394
2.0	3.1416	.021816	**5.0**	19.6350	.136354	**8.0**	50.2655	.349066
.1	3.4636	.024053	.1	20.4282	.141863	.1	51.5300	.357847
.2	3.8013	.026398	.2	21.2372	.147480	.2	52.8102	.366737
.25	3.9761	.027612	.25	21.6475	.150330	.25	53.4562	.371223
.3	4.1548	.028852	.3	22.0618	.153207	.3	54.1061	.375736
.4	4.5239	.031416	.4	22.9022	.159043	.4	55.4177	.384845
.5	4.9087	.034088	.5	23.7583	.164988	.5	56.7450	.394063
.6	5.3093	.036870	.6	24.6301	.171042	.6	58.0880	.403389
.7	5.7256	.039760	.7	25.5167	.177205	.7	59.4468	.412825
.75	5.9396	.041247	.75	25.9672	.180328	.75	60.1320	.417584
.8	6.1575	.042760	.8	26.4208	.183477	.8	60.8212	.422370
.9	6.6052	.045869	.9	27.3397	.189859	.9	62.2114	.432024
3.0	7.0686	.049087	**6.0**	28.2743	.196350	**9.0**	63.6173	.441786
.1	7.5477	.052414	.1	29.2247	.202949	.1	65.0388	.451658
.2	8.0425	.055851	.2	30.1907	.209658	.2	66.4761	.461640
.25	8.2958	.057609	.25	30.6796	.213053	.25	67.2006	.466671
.3	8.5530	.059396	.3	31.1725	.216475	.3	67.9291	.471730
.4	9.0792	.063050	.4	32.1699	.223402	.4	69.3978	.481929
.5	9.6211	.066813	.5	33.1831	.230438	.5	70.8822	.492237
.6	10.1788	.070686	.6	34.2119	.237583	.6	72.3823	.502654
.7	10.7521	.074667	.7	35.2565	.244837	.7	73.8981	.513181
.75	11.0447	.076699	.75	35.7847	.248505	.75	74.6619	.518486
.8	11.3411	.078758	.8	36.3168	.252200	.8	75.4296	.523817
.9	11.9459	.082958	.9	37.3928	.259672	.9	76.9769	.534561

The above table may be used for finding the areas of circles whose diameters are not within the limits of the table. Since the areas vary as the squares of their diameters, the given diameter may be divided (or multiplied) by 10, and the area found from the table under the resulting diameter corrected by moving the decimal point two places to the right (or left). Thus to find the area of a 22-inch circle:

From table, area of 2.2-inch circle = 3.8013 sq. in. = .026398 sq. ft.
Therefore area of 22-inch circle = 380.13 sq. in. = 2.64 sq. ft.

Again, to find the area of a 0.75-inch circle:

From table, area of 7.5-inch circle = 44.1786 sq. in. = 0.306796 sq. ft.
Therefore area of 0.75-inch circle = 0.4418 sq. in. = 0.00307 sq. ft.

It will also be apparent that the *first two* columns in the table may be used for any unit of measure.

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